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LARGE DEFLECTION OF ELASTIC PLATES UNDER PATCH LOADING

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INTRODUCTION

Most thin plate components of plated structures exhibit some degree of large deflection behavior under working conditions when subjected to transverse loading. In a recent work (1) it is shown that small deflection theory (classical theory of plates) may overestimate the deflection and stresses of thin plates up to 100 % for a number of practical problems, and can lead to an unduly conservative design. The large deflection theory allows for the stiffening effects due to membrane stresses which develop under finite deflections.

In some cases, such as ship and aircraft structures, an elastic large deflection theory is of special interest, for the plate components undergo deflections of the order of their thickness before the onset of plasticity. Through the use of newly developed high strength synthetic materials, expectations are that thinner and lighter plates will find increased application, especially in flight structures, thus posing a higher demand on large deflection theories.

The function of a thin plate element is generally to withstand a distributed lateral pressure, or to act with the adjoining structure in sustaining inplane forces, or both. It is, however, frequent that a plate may be subjected to a concentrated lateral loading under working conditions, such as the wheel loading of a forklift truck on decks of a cargo ship. In such a case, it is desired to confine the stresses under the wheels to permissible values in order to avoid excessive deformations and permanent set due to local yielding.

Although the equations of the large deflection behavior of plates were first derived in 1910 (18), it is only through recent advances in the development of numerical methods that the general problem of plates have been treated satisfactorily. The early investigators used the Ritz energy method (20) infinite

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double Fourier series (9) and finite differences (8,19) to obtain solutions for a number of isolated plates under uniformly distributed loading and with simple boundary conditions. Using finite differences and modern computational techniques, Basu (5) restudied the problem and obtained several solutions for new boundary conditions. The writer made an analysis of the general large deflection behavior of plates with various boundary conditions using a refined finite difference formulation and solution procedure (2). The application of the problem to the particular field of ship construction was later reported in Ref. 4.

The emergence of the Dynamic Relaxation method (13), and its development into a general numerical technique for the solution of partial differential equations (14), lead to a renewed attempt for solving the complex nonlinear large deflection equations of plates (15,22).

As part of the general development of finite element techniques, a number of diversified treatments have been proposed (10,12,23) for the large deflection behavior of plates. In this field the large deflection solutions have not so far reached a close agreement with one another and with the other established approaches. The discrepancies require yet more research to be resolved satisfactorily.

There is no general solution available yet, for the elastic large deflection behavior of plates under patch loading. The only large deflection solutions reported in the literature for this type of problem appear to be a particular solution in Ref. 2 and several solutions due to Weiss (21), who, using finite differences and a uniform mesh of 4 × 4 in 1/4 of the plate, treats the case of plates under central patch loading with specific elastic boundary conditions.

Herein, the solutions of plates under patch loading are generalized and a set of general solutions applicable to a wide range of plates are offered. The solutions offer significant improvements to the previous works in respect of generality (provide for a wide range of plate dimensions, magnitudes of loading, patch sizes and boundary conditions), accuracy (use of fine mesh), and correct representations of patch sizes (fine and graded mesh). The solutions lead to certain conclusions in regards to stresses under loading, which are characteristic of the large deflection behavior of plates and are of particular value in design of plates.

Furthermore, two methods are presented herein, which are developed to a highly refined stage for the analysis of the large deflection behavior of plates. These are based on: (1) Direct finite differences approach, and (2) the dynamic relaxation method. The treatment of the problem by the widely used finite element method is also outlined. Solutions from the three methods are compared and their merits pointed out.

The present analysis considers the behavior of an isolated square plate supported on four sides. Note that practical plate components do not act in isolation. Together with the adjoining structural elements they resist the applied loading. For a realistic estimate of stresses in a practical plate, the boundary conditions of the isolated plate problem considered need be carefully studied and formulated such as to allow for the influence of boundary continuity which exists in the actual structure. The solutions offered herein relate to two different cases of boundary conditions, which are chosen such as to exhibit a lower and an upper bound degree of restraint to the plate boundaries in comparison to the restraint exerted by the adjoining structures. The conditions present in an actual structure fall within the two cases analyzed. The solutions obtained are found to be fairly close to one another and predict with reasonable accuracy the final stresses of an actual structure.

FORMULATION AND SOLUTION

The basic concepts of three methods for the solution of the large deflection behavior of plates together with a description of the boundary conditions of the cases examined are summarized in the following.

Formulation 1.—The elastic large deflection behavior of rectangular isotropic plates, in which deformations are finite but slopes are assumed to be

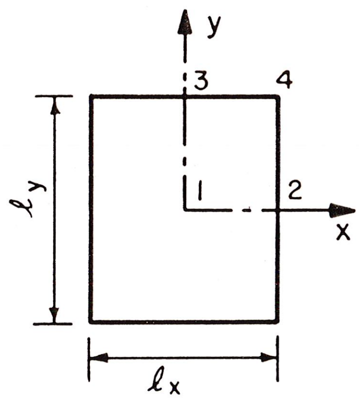

FIG. 1.—COORDINATE SYSTEM

small, may be expressed by the following two simultaneous fourth order partial differential equations due to von Karman (18):

$$(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) - (f_{,yy}\, w_{,xx} - f_{,xy}\, w_{,xy} + f_{,xx}\, w_{,yy}) = \frac{q}{D} \quad \text{(1)}$$

$$(f_{,xxxx} + 2f_{,xxyy} + f_{,yyyy}) = Eh\left[(w_{,xy})^2 - w_{,xx}\, w_{,yy}\right] \quad \text{(2)}$$

in which the subscripts after the comma each imply differentiation of the function with respect to that variable. Variable f is the Airy's stress function and is defined such that

$$N_x = f_{,yy},\ N_y = f_{,xx},\ N_{xy} = -f_{,xy} \quad \text{(3)}$$

All the other quantities related to the bending and membrane actions of the plate can be expressed in terms of the two variables w and f (17).

The boundary conditions considered for $x = l_x/2$ (Fig. 1) are as follows:

For bending action

1. Boundary on rigid supports

$$w = 0 \quad \text{(4)}$$

2. For rotationally free cases (shown symbolically by single lines on the diagrams)

$$w_{,xx} = 0 \qquad (5)$$

For rotationally fixed cases (shown symbolically by double lines on the diagrams)

$$w_{,x} = 0 \qquad (6)$$

For membrane action two extreme cases of fully free or fully restrained are considered.

3. For fully free condition zero membrane direct stress

$$f_{,yy} = 0 \qquad (7)$$

Zero membrane shear stress

$$f_{,xy} = 0 \qquad (8)$$

Eqs. 7 and 8 refer to the boundary conditions of an isolated free plate.

4. For fully restrained condition extensional displacement of the edge, u, is zero, thus

$$u = \int_0^{l_x/2} \epsilon_x dx = \frac{1}{Eh} \int_0^{l_x/2} (f_{,yy} - \nu f_{,xx})\, dx$$

$$- \frac{1}{2} \int_0^{l_x/2} (w_{,x})^2\, dx = 0 \qquad (9)$$

tangential displacement is zero $v = 0$ . . . (10)

For solution, the plate is subdivided into a graded mesh. The grading is chosen to be finer under the central patch loading and the fixed boundary conditions for increased accuracy (for mesh details see Appendix I). Eqs. 1 and 2, together with the boundary conditions are expressed at the nodes in terms of central finite difference expressions duly modified for a graded mesh (1). The resulting equations may then be arrayed into the following coupled matrix form

$$[A(f)]\, \{w\} = \{p\} \qquad (11)$$

$$[B]\, \{f\} = \{r(w)\} \qquad (12)$$

in which $[A(f)]$ = a square matrix depending on f; $\{w\}$ and $\{f\}$ = column matrices of the unknown variables w and f, respectively; $[B]$ = a square matrix with constant coefficients obtained from the left-hand side of Eq. 2; and $\{p\}$ is the applied transverse loading. Matrix $\{r(w)\}$ is a column matrix depending on w and representing the right-hand side of Eq. 2 for specified values of w. For numerical evaluation, it is convenient to rewrite Eq. 12 as follows:

$$\{f\} = [B]^{-1}\{r(w)\} \quad (13)$$

Now for any specified condition, Eqs. 11 and 13 can be solved for w and f using an iterative procedure. There are various schemes used for the solution of the present type of coupled equations (1,5,19,21). In principle they all use the same concept and converge fairly well (2-6 iterations per loading). The solution is obtained by successive approximations as follows. For a given applied loading $\{p\}$, $[A(f)]$ is evaluated by assigning a value to f (may be assumed equal to zero for first loading, and equal to the previous values for subsequent loadings). Thus Eq. 11 can be solved for $\{w\}$, from which $\{r(w)\}$ is evaluated, and subsequently used in Eq. 13 to give new values of $\{f\}$. Term $[A(f)]$ can now be recalculated using the newly obtained values of f. The procedure is repeated until a desired degree of accuracy is reached. There are certain refinements in the iteration which are employed to ensure a rapid convergence (for details of these refinements the previously quoted references should be consulted).

Formulation 2.—An alternative finite difference approach to the above problem is the Dynamic Relaxation method which has been recently developed by Otter (13), and is employed to analyze a number of plate and shell problems (14,16,22). For the case of plates, instead of solving Eqs. 1 and 2 directly, the equivalent dynamic problem of its damped vibrations may be considered. The dynamic problem can be solved by an explicit finite difference method, which involves a simple substitution routine on a digital computer and avoids the solution of the simultaneous matrix Eqs. 11 and 13, thus resulting in tremendous computer storage savings. Provided that the oscillations of the dynamic problem are damped so that they rapidly die out, a satisfactory solution to the static finite difference equations is obtained. The method, being a finite difference approach, retains all the generality features of the previous formulation in treatment of mixed boundary conditions and nonuniform loading. For the particular problem of the elastic large deflection analysis of rectangular plates, the equations of motion of a plate element (dxdy) can be written for motions along the three axes z, x and y in the following form for computational convenience:

$$(M_{x,xx} - 2M_{xy,xy} + M_{y,yy}) + (N_x\, w_{,xx} + 2N_{xy}\, w_{xy} + N_y\, w_{yy}) = \rho_w\, \dot{w}_{,t} - K_w\, \dot{w} - q \quad (14a)$$

$$N_{x,x} + N_{xy,y} = \rho_u\, \dot{u}_{,t} + K_u\, \dot{u} \quad (14b)$$

$$N_{y,y} + N_{xy,x} = \rho_v\, \dot{v}_{,t} + K_v\, \dot{v} \quad (14c)$$

in which $(\cdot)$ represents differentiation with respect to time t.

To each equation a viscous damping term is added. As the static solution is sought, mass densities need not represent the true values of the related material. It is found that for rapid convergence fictitious suitable mass densities may be evaluated and used (12).

The moments and membrane forces are expressed in terms of w, u and v by:

$$M_x = -D(w_{,xx} + \nu w_{,yy}) \quad (15a)$$

$$M_y = - D(w_{,yy} + \nu w_{,xx}) \quad \text{(15b)}$$

$$M_{xy} = D(1 - \nu)\, w_{,xy} \quad \text{(15c)}$$

$$N_x = \frac{Eh}{(1 - \nu^2)} \left\{ \left[u_{,x} + \frac{1}{2}(w_{,x})^2 \right] + \nu \left[v_{,y} + \frac{1}{2}(w_{,y})^2 \right] \right\} \quad \text{(16a)}$$

$$N_y = \frac{Eh}{(1 - \nu^2)} \left\{ \left[v_{,y} + \frac{1}{2}(w_{,y})^2 \right] + \nu \left[u_{,x} + \frac{1}{2}(w_{,x})^2 \right] \right\} \quad \text{(16b)}$$

$$N_{xy} = \frac{Eh}{2(1 + \nu)} (u_{,y} + v_{,x} + w_{,x}\, w_{,y}) \quad \text{(16c)}$$

The dynamic relaxation approach permits the use of the large deflection equations in the expanded form of Eqs. 14-16. This is so because in the dynamic relaxation method it is not required to reduce the number of final unknowns to a minimum as in formulation 1, for ease in obtaining a solution. The present form of formulation can be readily modified to cover plates of nonuniform thickness and orthotropic properties.

The bending boundary conditions are expressed in the same manner as formulation 1. For membrane boundary conditions u and v or membrane forces as given by Eq. 16 are equated to zero to achieve the conditions outlined in the previous formulation. To obtain a solution, the plate is subdivided by a mesh, and Eqs. 14-16 together with the boundary conditions are expressed in terms of finite difference expressions. The velocities between the two time intervals $[n - (1/2)]\,\Delta t$ and $[n + (1/2)]\,\Delta t$ are written in finite difference form as

$$w_{[n+(1/2)]\Delta t} = w_{[n-(1/2)]\Delta t} + \Delta t\, \dot{w}_{n\Delta t} \quad \text{(17a)}$$

$$u_{[n+(1/2)]\Delta t} = u_{[n-(1/2)]\Delta t} + \Delta t\, \dot{u}_{n\Delta t} \quad \text{(17b)}$$

$$v_{[n+(1/2)]\Delta t} = v_{[n-(1/2)]\Delta t} + \Delta t\, \dot{v}_{n\Delta t} \quad \text{(17c)}$$

in which n = the number of time intervals from datum.

The dynamic relaxation iteration is carried out by scanning the nodes over the plate successively and repeatedly. Each scanning represents a lapse of time Δt. At any time interval and at each node the following computations are made:

1. At a time $[n - (1/2)]\,\Delta t$, Eqs. 14 are used to calculate the velocity at time $n\Delta t$. This requires information obtained from previous calculations about the velocity, moments and forces at previous intervals.
2. These velocities are substituted into Eq. 17 to give displacement values at time $[n + (1/2)]\,\Delta t$.
3. The moments and forces at time $[n + (1/2)]\,\Delta t$ are then evaluated using the preceding displacements and Eqs. 15 and 16.

Now, at the subsequent time interval (next scan), the aforementioned procedure is repeated and the new displacements obtained are compared with the previous values. The iterations are continued until the desired accuracy is reached. Provided the damping coefficients chosen are below a critical value the process is convergent. The speed of convergence depends upon the values of the damping factors, K, mass densities ρ, mesh size and the time intercal Δt. The influence of each of these parameters is covered to some extent in Refs. 14 and 15. For the type of problem under consideration and for an accuracy specification of 0.5 %, the solutions take 80 cycles to 200 cycles to converge for each loading (i.e., the absolute difference between values of deflections in two successive iterations being less than 0.5 % of the last calculated deflection). The damping coefficients are chosen for slightly under damped oscillations, thus making the accuracy check possible.

The solutions obtained using the preceding outlined procedure are shown in Ref. 3, and through additional solutions obtained by the writer to agree closely with the formulation 1 for the large deflection elastic behavior of plates.

Formulation 3.—Much progress has been made in the nonlinear analysis of plates and shells as part of the general development of the finite element method. Both material and geometrical nonlinearities are treated. Recent progress has been mainly in the displacement method of finite element analysis. The techniques developed in this field are not as unified in formulation of nonlinear problems as the preceding formulations 1 and 2. The approaches are common in most of their algorithm, and reduce basically to the solution of a number of piecewise linear steps each consisting of a load increment. In some of the methods an iterative or a correction procedure is used at each load level. There are not yet sufficient data available in the literature for the geometrical nonlinear analysis of plates based on the various approaches, so that a critical comparison could be made to reveal the merits of each method and the significance of the refinements each claim. A general recent review of the finite element methods on the material and geometrical nonlinear problems of plates and shells is given in Ref. 6.

The formulations may be broadly divided into a current coordinate (Eulerian) and an initial coordinate (Lagrangian) approaches, although some investigators have used combinations of both. Based on the first approach Murray, et al. (12), and Zienkiewics (23) have presented formulations for the treatment of the problem. Murray expresses the element deformations in terms of a set of local coordinates which rotate and translate with the element, hence slopes are assumed negligible within each element. A square simply supported plate under a uniformly distributed load is case studied. Deflections obtained are in good agreement with the available differential equation formulations, but stresses are off up to 10 %.

Full finite strain formulation equations in a Lagrangian frame of reference is given by Hibbitt, et al. in Ref. 7. In Ref. 6 solutions for the square plate of the previous case are obtained from the Lagrangian formulation and are compared with solutions of other formulations. For the particular cases compared, Murray's initial coordinate approach (12) yields more accurate results.

SQUARE PLATES UNDER PATCH LOADING

Solutions are offered in a condensed and general form for square plates with the following two extreme boundary conditions:

1. Case A—Simply supported (rigidly supported, rotationally free) with

zero membrane shear and membrane direct stresses at the boundaries (Eqs. 4, 5, 7, and 8 as boundary conditions). This case corresponds to an isolated square plate under loading.

TABLE 1.—SIMPLY SUPPORTED PLATES UNDER CENTRAL PATCH LOADING—CASE A1-A4

Patch Size (1)	P' (2)	W_1 (3)	M'_{x1} (4)	N'_{x1} (5)	N'_{y2} (6)
Case A1	(Linear) 1	0.0443	1.72	0	0
α = 1	10	0.428	16.4	2.06	-3.88
β = 1	20	0.793	29.4	6.49	-12.7
	40	1.34	45.6	17.6	-36.7
	60	1.73	55.1	28.1	-62.1
	80	2.05	61.4	37.6	-86.9
	100	2.33	66.1	46.2	-11.1
Case A2	(Linear) 1	0.125	10.4	0	0
α = 0.1	10	1.02	87.9	17.0	-18.8
β = 0.1	20	1.61	147	42.2	-46.1
	40	2.37	232	90.4	-96.8
	60	2.91	299	135	-142
	80	3.35	356	177	-184
	100	3.72	407	217	-222
Case A3	(Linear) 1	0.121	7.92	0	0
α = 0.2	10	0.997	65.0	15.3	-18.5
β = 0.2	20	1.59	103	37.7	-46.4
	40	2.34	151	78.7	-99.0
	60	2.87	184	115	-146
	80	3.29	210	147	-189
	100	3.65	233	177	-229
Case A4	(Linear) 1	0.113	6.23	0	0
α = 0.3					
β = 0.3	10	0.941	50.8	12.7	-17.4
	20	1.52	79.3	31.0	-44.1
	40	2.24	112	64.3	-95.8
	60	2.75	133	92.7	-142
	80	3.15	149	118	-185
	100	3.51	162	142	-225

2. Case B—Clamped (rigidly supported, rotationally fixed) with zero membrane shear and zero extensional displacement (Eqs. 4, 6, 8, and 9 as boundary conditions). In Ref. 5 it is shown that the condition of zero extensional displacement results in a negligible tangential displacement at the boundaries, and may be considered to represent closely the case of a plate welded to an infinitely rigid surrounding.

For each case central patch loadings with 5 different dimensions are considered ($\alpha \times \beta$ being 0.1 × 0.1, 0.2 × 0.2, 0.3 × 0.3, 0.2 × 0.3, and 0.2 × 0.4). The loading is assumed to act uniformly over the patch area. A sixth condition of uniformly distributed load over the whole plate ($\alpha \times \beta$ = 1 × 1) is included for each case for comparison. Each solution is obtained for increasing values of patch loading parameter $P' = l_x^2 P/(h^4 E)$ using formulation 1 and 10 equal load increments. Graded meshes are used with details as given in the Appendix I.

The results are summarized in Figs. 2 to 5 which show the general large deflection behavior of the problem, and in Tables 1 to 4 which give the numerical values of maximum deflections and stresses. The deflections and stresses are presented in a nondimensional form for a Poisson's ratio of 0.3. The numerical suffices in the figures and the tables refer to the location in the plate (as shown in Fig. 1) to which the values refer.

TABLE 2.—SIMPLY SUPPORTED PLATES UNDER CENTRAL PATCH LOADING—CASES A5, A6

Patch Size (1)	P' (2)	W_1 (3)	M'_{x1} (4)	M'_{y1} (5)	N'_{y1} (6)	N'_{y2} (7)	N'_{x1} (8)	N'_{x3} (9)
Case A5	(Linear) 1	0.117	7.20	6.75	0	0	0	0
α = 0.2	10	0.968	59.1	54.8	14.3	-17.8	13.5	-18.2
β = 0.3	20	1.55	93.8	85.6	35.1	-44.8	32.9	-45.8
	40	2.99	136	121	73.3	-96.2	67.8	-98.8
	60	2.81	165	145	107	-142	97.6	-147
	80	3.22	188	163	137	-184	124	-190
	100	3.58	208	178	165	-223	149	-231
Case A6	(Linear) 1	0.112	6.59	5.88	0	0	0	0
α = 0.2	10	0.936	54.5	47.8	13.1	-16.8	11.8	-17.6
β = 0.4	20	1.51	86.5	73.9	32.4	-42.8	28.8	-44.9
	40	2.23	125	103	68.2	-92.5	58.9	-98.3
	60	2.74	152	121	99.3	-137	84.1	-147
	80	3.15	173	135	127	-178	106	-192
	100	3.49	191	146	154	-216	126	-233

The nondimensionalization is carried out such that for assumed values of concentrated loading and plate dimensions, the corresponding deflections, and bending and membrane stresses may be readily evaluated for a range of patch sizes. In the previous treatment (21) a priori knowledge of contact area has been necessary for utilizing the results.

The relationship between the nondimensional coefficients (W, P', N'_x, N'_y, M'_x, M'_y) and plate deflection and stresses are as follows [The nondimensionalization coefficients chosen extend the application of results to certain families of transformed orthotropic plates (3)]:

$$P = \left(\frac{h^4 E}{l_x^2}\right) P'; \quad w = (h)\, W; \quad \sigma_{xm} = \left(\frac{D}{l_x^2 h}\right) N_x' = \left(\frac{h^2 E}{10.92\, l_x^2}\right) N_x';$$

$$\sigma_{xb} = \left(\frac{6}{h^2}\,\frac{D}{l_x^2}\sqrt{\frac{D}{Eh}}\right) M_x' = \left(\frac{h^2 E}{6.015\, l_x^2}\right) M_x';$$

$$\sigma_{ym} = \left(\frac{D}{l_x^2 h}\right) N_y' = \left(\frac{h^2 E}{10.92\, l_x^2}\right) N_y';$$

$$\sigma_{yb} = \left(\frac{6}{h^2}\,\frac{D}{l_x^2}\sqrt{\frac{D}{Eh}}\right) M_y' = \left(\frac{h^2 E}{6.015\, l_x^2}\right) M_y' \quad \ldots\ldots\ldots\ldots\ldots \quad (18)$$

TABLE 3.—CLAMPED PLATES UNDER CENTRAL PATCH LOADING—CASES B1-B4

Patch Size (1)	P' (2)	W_1 (3)	M'_{x1} (4)	M'_{x2} (5)	N'_{y1} (6)	N'_{y2} (7)	N'_{y3} (8)	N'_{y4} (9)
Case B1	(Linear) 1	0.0137	0.816	-1.85	0	0	0	0
$\alpha = 1$	10	0.136	8.06	-18.4	0.650	-0.021	0.462	0.207
$\beta = 1$	20	0.266	15.6	-36.3	2.44	-0.075	1.74	0.771
	40	0.491	28.1	-69.3	8.26	-0.220	6.03	2.65
	60	0.673	37.3	-98.6	15.4	-0.334	11.5	4.98
	80	0.821	44.2	-125	22.8	-0.380	17.4	7.43
	100	0.946	49.6	-149	30.2	-0.352	23.5	9.89
Case B2	(Linear) 1	0.0605	8.43	-4.45	0	0	0	0
$\alpha = 0.1$	10	0.536	77.1	-40.1	12.6	-1.08	4.95	4.41
$\beta = 0.1$	20	0.899	136	-69.1	35.1	-2.78	13.6	11.7
	40	1.36	225	-110	82.4	-5.67	31.1	25.5
	60	1.68	290	-143	127	-7.85	47.4	37.4
	80	1.92	350	-170	170	-9.56	62.4	47.8
	100	2.15	398	-195	212	-11.0	76.5	57.3
Case B3	(Linear) 1	0.0568	6.01	-4.45	0	0	0	0
$\alpha = 0.2$	10	0.507	53.9	-40.5	11.0	-1.02	4.52	3.23
$\beta = 0.2$	20	0.854	91.6	-70.4	30.5	-2.87	12.5	8.98
	40	1.28	141	-113	70.4	-6.73	28.5	20.8
	60	1.60	175	-147	107	-10.4	42.9	31.8
	80	1.83	201	-175	140	-13.8	56.1	42.1
	100	2.03	224	-200	172	-17.2	68.3	51.9
Case B4	(Linear) 1	0.0505	4.38	-4.26	0	0	0	0
$\alpha = 0.3$	10	0.457	39.4	-39.3	8.57	-0.656	4.26	2.72
$\beta = 0.3$	20	0.782	66.6	-69.2	23.8	-1.80	12.0	7.62
	40	1.20	100	-114	55.7	-4.10	28.5	18.0
	60	1.48	122	-148	84.6	-6.10	43.9	27.5
	80	1.70	138	-177	111	-7.85	58.2	36.2
	100	1.89	152	-204	135	-9.42	71.6	44.3

In Tables 1-4, the first row given for each case is the linear small deflection coefficients obtained as part of the general large deflection solutions. The small deflection solutions given by Timoshenko for square plates under partial loading (Ref. 17, Table 19) agree closely with the coefficients of the present analysis (for the smallest patch size agreement to within 1.5 %).

Fig. 2 shows the central deflection of simply supported and clamped cases for increasing values of loading. It is interesting to note that for given values of patch loading, the difference between the central deflections of the two conditions $\alpha \times \beta = 0.1 \times 0.1$ and 0.2×0.2 is a few percents for the same loading (depending on the magnitude of loading). The results suggest that, for deflections, it is unnecessary to have an accurate knowledge of the wheel contact area. The deflection profiles of the cases analyzed are shown in Fig. 3 for the maximum values of loading considered. Fig. 4 shows the bending moment profiles of the plates for the maximum loading. The bending moments along the center lines are normalized for ease of comparison and are shown

TABLE 4.—CLAMPED LOADING UNDER CENTRAL PATCH LOADING—CASES B5, B6

Patch Size (1)	P' (2)	W_1 (3)	M'_{x1} (4)	M'_{x2} (5)	M'_{y1} (6)	M'_{y3} (7)	N'_{y1} (8)	N'_{y2} (9)	N'_{x1} (10)	N'_{x3} (11)	N'_{x2} (12)	N'_{x4} (13)
Case 5B	(Linear) 1	0.0535	5.31	-4.24	4.88	-4.47	0	0	0	0	0	0
$\alpha = 0.2$	10	0.482	47.9	-38.9	43.7	-41.1	9.71	-7.40	9.38	-8.50	4.43	2.95
$\beta = 0.3$	20	0.817	81.5	-61.2	73.5	-72.2	27.3	-2.05	26.2	-2.39	12.5	8.29
	40	1.25	125	-110	111	-117	63.5	-4.66	60.7	-5.55	29.3	19.3
	60	1.54	155	-143	135	-153	96.7	-6.98	91.8	-8.43	44.7	29.5
	80	1.77	178	-171	154	-183	127	-9.07	120	-11.1	59.0	38.9
	100	1.96	198	-196	169	-210	156	-11.0	146	-13.5	72.4	47.7
Case B6	(Linear) 1	0.0499	4.73	-4.00	4.01	-4.50	0	0	0	0	0	0
$\alpha = 0.2$	10	0.453	42.9	-37.0	36.5	-41.8	8.58	-0.555	8.15	-0.803	4.25	2.83
$\beta = 0.4$	20	0.777	73.6	-65.3	61.4	-47.4	24.1	-1.51	22.7	-2.27	12.0	8.00
	40	1.19	113	-106	91.4	-123	56.8	-3.37	52.9	-5.40	28.8	19.0
	60	1.48	140	-138	110	-160	86.9	-4.93	80.0	-8.26	44.5	29.3
	80	1.70	162	-165	125	-193	115	-6.27	105	-10.9	59.1	39.7
	100	1.88	180	-189	136	-222	140	-7.46	127	-13.3	72.8	47.5

in the figure as ratio of the moment to its value at the plate center, M_{x1}, in each case. The figure shows that, for smaller patch dimensions moments concentrate at plate center. For the actual values of moments Tables 1-4 should be consulted. The distribution of membrane stresses along and across the center line for the same loading is given in Fig. 5. Note that for higher values of loading and small patch sizes (e.g., 0.1×0.1), the magnitudes of bending moments and membrane forces for the two cases of simply supported and clamped conditions are each within 3 % of one another (Tables 1 and 3) at center. It may be concluded that at higher values of loading, the stresses under a patch loading are not influenced significantly by the plate's outer boundary conditions and a rigorous analysis for the evaluation of boundary restraints under practical conditions may not be justified. The two extreme solutions given predict with reasonable accuracy the final stresses in a practical plate.

The solutions given by Weiss (21) for square plates under patch loading

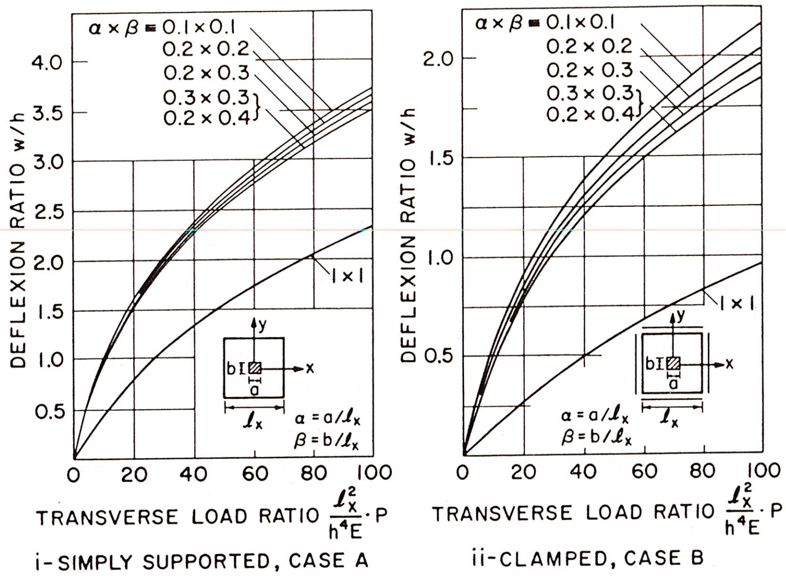

FIG. 2.—PLATES UNDER CENTRAL PATCH LOADING—VARIATION OF CENTRAL DEFLECTION WITH LOADING FOR VARIOUS PATCH DIMENSIONS

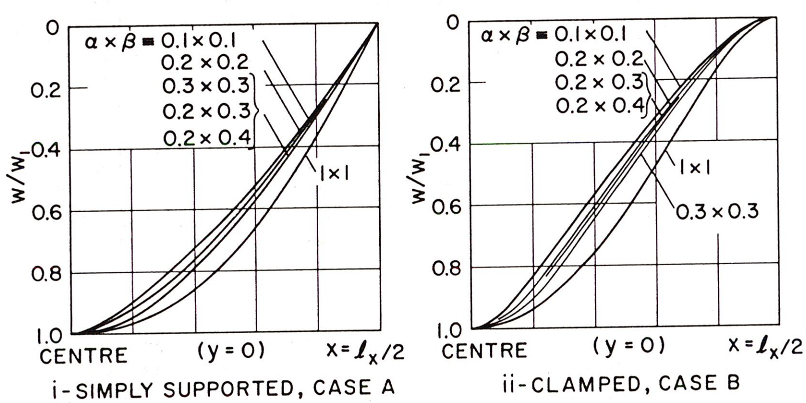

FIG. 3.—PLATES UNDER CENTRAL PATCH LOADING (P' = 100)—DEFLECTION PROFILES ALONG y = 0 CENTER LINE

and with elastic boundary conditions fall within the corresponding two extreme values given in the Tables 1-4. Results of Ref. 21, however, do not extend over the entire load range covered in the present analysis.

NUMERICAL EXAMPLE

To illustrate the use of the results for the analysis of a specific problem, consider a square plate with the following dimensions and material proper-

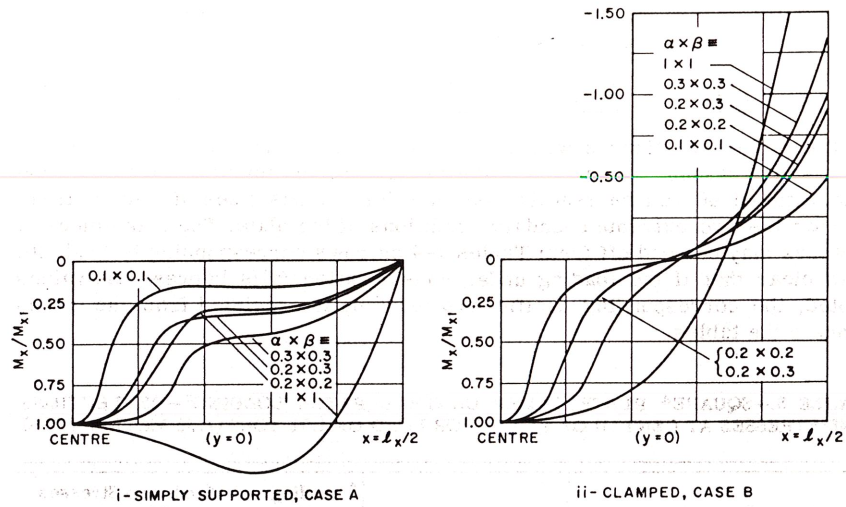

FIG. 4.—PLATES UNDER CENTRAL PATCH LOADING (P' = 100)—BENDING MOMENT PROFILES ALONG y = 0 CENTER LINE

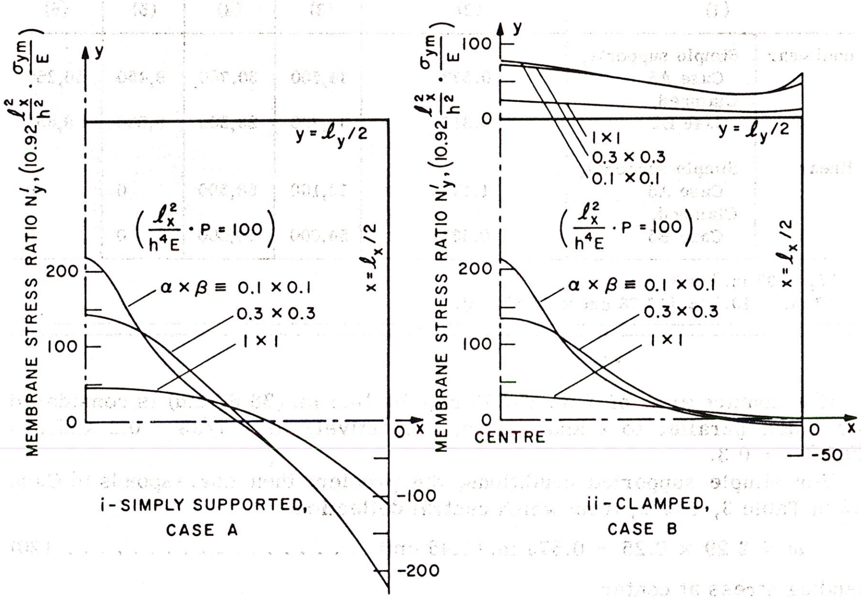

FIG. 5.—PLATES UNDER CENTRAL PATCH LOADING, (P' = 100)—DISTRIBUTION OF MEMBRANE STRESS N'_y ACROSS THE CENTER LINES AND THE BOUNDARIES

ties, subjected to a central load of 1.71 tons, (1.55 metric tons): l_x = 35 in. (88.9 cm); h = 0.25 in. (0.635 cm); E = 30 × 10⁶ psi (2.1 × 10⁶ kgf/cm²); and ν = 0.3. Therefore, the nondimensional value of loading

$$P' = \frac{l_x^2}{h^4 E} P = \frac{35^2 \times 1.71 \times 2240}{0.25^4 \times 30 \times 10^6} = 40 \quad \text{(19)}$$

The tables and figures will now be consulted for the evaluation of deflections and stresses. From Fig. 2, the range of deflections for the transverse load ratio of 40 can be readily inspected for various sizes of contact areas and for the two extreme boundary conditions of the plate. The magnitudes of stresses may be read off from Tables 1-4 on rows corresponding to load, 40. It is clear that if the loading under consideration falls between the values quoted, the corresponding coefficients must be interpolated from the values given in the tables.

TABLE 5.—SQUARE[a] PLATE UNDER CENTRAL PATCH LOADING[b]—DEFLECTIONS AND STRESSES AT CENTER OF PLATE FOR LOAD OF 1.71 TONS (1.55 METRIC TONS)

Case (1)		Central deflection, w, in inches (2.54 cm) (2)	Bending and Membrane Stresses at Center, in pounds per square inch, (0.0703 kgf/cm²)			
			σ_{xb} (3)	σ_{yb} (4)	σ_{xm} (5)	σ_{ym} (6)
nonlinear	Simple supports, Case A5	0.573	34,500	30.700	9,450	10,250
	Clamped, Case B5	0.312	31,700	28,200	8,500	8,850
linear	Simple supports, Case A5	1.17	73,100	68,500	0	0
	Clamped, Case B5	0.535	54,000	48,000	0	0

[a] l_x = 35 in. (88.9 cm).
[b] 7 in. × 10.5 in. (17.78 cm × 26.67 cm).

If a contact area of 7 in. (17.78 cm) by 10.5 in. (26.67 cm) is considered for sides parallel to x and y-axes, respectively, α = 7/35 = 0.2 and β = 10.5/35 = 0.3.

For simple supported conditions, the problem then corresponds to Case A5 in Table 3, row 5, from which central deflection

$$w = 2.29 \times 0.25 = 0.573 \text{ in. (1.45 cm)} \quad \text{(20)}$$

bending stress at center

$$\left.\begin{aligned} \sigma_{xb} &= \frac{h^2 E}{6.015\, l_x^2} M'_{x1} \\ &= \frac{0.25^2 \times 30 \times 10^6}{35^2 \times 6.015} M'_{x1} = 254\, M'_{x1} \text{ psi (17.86 kgf/cm}^2) \end{aligned}\right\} \quad \text{(21)}$$

and membrane stress at center

$$\left.\begin{aligned}\sigma_{xm} &= \frac{h^2 E}{10.92\, l_x^2}\, N'_{x1}\\ &= \frac{0.25^2 \times 30 \times 10^6}{35^2 \times 10.92}\, N'_{x1} = 140\, N'_{x1}\ \text{psi}\ (9.84\ \text{kgf/cm}^2)\end{aligned}\right\} \quad \ldots\ldots (22)$$

Deflection and stresses at center of the plate for the two extreme boundary conditions of simply supported (Case A5) and clamped (Case B5) are summarized in Table 5 together with the small deflection linear solutions obtained from the same tables for comparison.

Note that the deflections and stresses given for the linear theory do not develop in reality, for the actual behavior of a plate is nonlinear. The linear values are given to indicate the order of error and the magnitudes which the small deflection theory yields. The comparison shows that in the given case, the deflections and stresses as calculated from the small deflection theory may be up to 100 % in error. Also note that, although the nondimensional value of loading chosen in the example is less than one-half of the load range covered, the corresponding stresses of the two upper (Case A5) and lower (Case B5) limits obtained are at most 14 % different from one another for the large deflection behavior. The stresses of the two upper and lower restraint conditions are close enough to give a reasonable estimate of stresses in a practical plate element.

CONCLUSIONS

Two general methods based on: (1) The direct finite difference approach; and (2) the dynamic relaxation method are presented for the treatment of the elastic large deflection behavior of plates under transverse loading. The presentation is followed by an outline of the main aspects of the widely used finite element method for the analysis of the same problem. Using the preceding three methods, the problem of the large deflection behavior of square plates under a central patch loading is studied.

The direct finite difference and the dynamic relaxation methods use basically the same partial differential equations, and for the same geometry and mesh, they yield the same results both for deflections and stresses. The solutions agree well with the results of the previous investigators where available. In Ref. 3, a comparison between the solutions obtained from the two approaches shows that for a transverse load of $P' = 688$ spread uniformly over the plate (type of Case A1) and causing a central deflection of $w/h = 5.9$, the agreement between the two results is as follows: for central deflection within 2.5 %, for central moments 4 %, and for membrane forces 3.7 %. The two methods are best employed in the analysis of circular and rectangular plates, where the governing differential equations may be written without difficulty. The finite element method has the strong feature of being applicable with equal ease to plates of irregular shapes. With respect to accuracy in the analysis of large deflection of plates, the various formulations of the finite element techniques are found to differ up to 10 % in deflections and more in stresses (6). For the case of a central patch loading of $\alpha \times \beta = 0.2 \times 0.2$ the finite element formulation developed by Marcal, et al. (7) and incorporated in a general purpose program (11) was employed for investigation.

Taking advantage of symmetry, a 16 triangular element mesh was used for 1/8 of the plate to analyze both the simply supported condition (Case A3) and the clamped condition (Case B3). The loading increment was chosen the same as formulations 1 and 2. For maximum loading considered, deflections obtained for the simply supported condition were 15 %, and the clamped condition 8.4 % higher than the previous formulations. The error in stresses is higher, especially for membrane stresses at lower stages of loading. The load deflection curves obtained from the finite element and the finite difference solutions are given in Fig. 6 for comparison.

The computer time requirement in each case depends on the computer system and the particular programming details. An accurate general statement for the time requirement would be difficult, but the experience obtained indicates that the order of time required on a digital computer for each complete solution of the type presented would be three time units for formulation 1, five time units for formulation 2 and 40 time units for formulation 3. The reasons for the slightly longer computational time required in formulation 2 (dynamic relaxation) are that: (1) Convergence becomes consistently slower with increase in mesh fineness; and (2) the initial values of the optimum damping coefficients evaluated become less effective with increase in deflection and development of membrane forces. To speed convergence reevaluation of damping coefficients has been necessary.

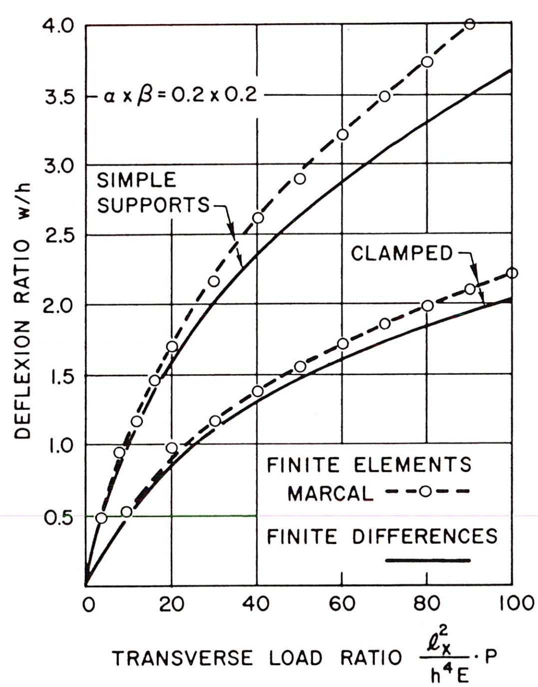

FIG. 6.—PLATES UNDER CENTRAL PATCH LOADING $\alpha \times \beta = 0.2 \times 0.2$—COMPARISON OF VARIATION OF CENTRAL DEFLECTION WITH LOADING BETWEEN FINITE ELEMENT AND FINITE DIFFERENCE SOLUTIONS

The computational storage requirement for each formulation varies with the size of the problem. Direct finite difference and the finite element formulations require about the same computational storage. For the problems case studied herein, the storage requirement of the dynamic relaxation method is of the order of 10 % of the other two methods. The savings in storage would be more for larger problems.

Two series of square plates with simply supported and clamped conditions are analyzed. The results, as summarized in the Tables 1-4, illustrate the significance of large deflection results compared to the small deflection solutions both in deformations and stresses. For example, for the maximum loading considered ($P' = 100$) and for the patch load of 0.1×0.1, from Tables 1 and 3, the central deflections and maximum moment coefficients are given in Table 6. A comparison of the values of Table 6 and the numerical example given, shows the magnitudes of error in the small deflection solutions and

TABLE 6.—COMPARISON OF RESULTS FOR PATCH LOAD OF $P' = 100$ ON AREA $\alpha \times \beta = 0.1 \times 0.1$

Cases (1)		Central deflection W_1 (2)	Maximum moment M'_{x1} (3)
simply supported (Case A2)	linear	12.5	1,040
	nonlinear	3.72	407
clamped (Case B2)	linear	6.05	843
	nonlinear	2.15	398

makes the importance of the use of large deflection solutions in the stress analysis of plates in avoiding unduly conservative designs evident.

Furthermore, the results reveal that for the same loading and patch sizes, membrane and bending stresses under the patch loading of the two simply supported and clamped cases are within a few percents of one another. As the two cases act as lower and upper limits to the boundary restraint exerted on a loaded practical panel by the adjoining structure, the results lead to the simplifying suggestion that for engineering design purposes, so far as the plates are rigidly supported on the four sides, the nature of their continuity over the supports may be disregarded.

For central deflections, the solutions indicate that in each case for the same magnitude of loading, deflections vary slightly for close patch sizes but differ widely between the simply supported and the clamped cases.

The solutions offered are general, and apart from their contribution towards a better understanding of the general large deflection behavior of plates, they provide a useful background for the advanced design of thin plates under patch loading and lead to a more rational and economical design.

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APPENDIX I.—MESH SIZES

In obtaining the solutions, the mesh used in each case is graded such that divisions are finer under the loading and express the loading dimensions stipulated by α and β exactly. The grading of the meshes used for the solutions given are listed in Table 7.

TABLE 7.—GRADED MESH USED, (FIRST QUADRANT OF PLATE DIVIDED INTO AN 8 × 8 GRADED ORTHOGONAL MESH)

Patch size $\alpha \times \beta$ (1)	Axis (2)	Distances of mesh lines from center to side as ratios of $l_x/2$ (3)
1 × 1	x-x y-y	uniform
0.1 × 0.1	x-x y-y	0.0, 0.067, 0.133, 0.225, 0.375, 0.562, 0.750, 0.906, 1
0.2 × 0.2	x-x y-y	0.0, 0.133, 0.267, 0.392, 0.517, 0.656, 0.781, 0.906, 1
0.3 × 0.3	x-x y-y	0.0, 0.120, 0.240, 0.360, 0.485, 0.610, 0.781, 0.906, 1
0.2 × 0.3	x-x	same as 0.2 × 0.2
	y-y	same as 0.3 × 0.3
0.2 × 0.4	x-x	same as 0.2 × 0.2
	y-y	0.0, 0.144, 0.229, 0.343, 0.457, 0.582, 0.781, 0.906, 1

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APPENDIX III.—NOTATION

The following symbols are used in this paper:

$[A(f)]$ = coefficients of nodal values of w from Eq. 1;
a = length of loading in x direction;
$[B]$ = coefficients of nodal values of f from Eq. 2;
b = length of loading in y direction;
D = flexural stiffness of plate;
E = Young's modulus of elasticity;
f = Airy's stress function;
$\{f\}$ = nodal values of Airy's stress function;
h = plate thickness;
K_w, K_u, K_v = viscous damping coefficients in z, x and y directions;
l_x, l_y = plate sides in x and y directions;
M_x, M_y, M_{xy} = moment per unit length of plate;
M_x' = $6.015\ l_x^2\ \sigma_{xb}/(h^2 E)$;
M_y' = $6.015\ l_x^2\ \sigma_{yb}/(h^2 E)$;
N_x, N_y, N_{xy} = membrane forces per unit length of plate;
N_x' = $10.92\ l_x^2\ \sigma_{xm}/(h^2 E)$;
N_y' = $10.92\ l_x^2\ \sigma_{ym}/(h^2 E)$;
n = number of time increments Δt from datum;
P = magnitude of applied patch loading;
P' = $l_x^2 P/(h^4 E)$;
$\{p\}$ = nodal vector of applied loading in z direction;
q = intensity of transverse loading;
$\{r(w)\}$ = nodal values of w terms in Eq. 2;
t = time;
u, v, w = displacements in x, y and z directions;
W = w/h;
x, y, z = rectangular coordinates;
α = a/l_x;
β = b/l_x;
Δ = increment;
ϵ = strain;
ν = Poisson's ratio;
ρ_u, ρ_v, ρ_w = mass densities in x, y and z directions;
σ_x, σ_y = stresses in x and y directions, subscripts b and m refer to bending and membrane stresses; and
$(\cdot)$ = differentiation with respect to time.

9359 ELASTIC PLATES UNDER PATCH LOADING

KEY WORDS: **Concentrated loads; Deflection; Dynamic properties;** Elastic theory; **Finite differences; Loads (forces); Rectangular bodies; Relaxation (mechanics);** Stresses; **Structural engineering; Thin plates**

ABSTRACT: Two methods based on: (1) The direct finite difference approach; and (2) the dynamic relaxation method are presented for the treatment of the elastic large deflection behavior of plates under transverse loading. The merits of the methods in respect of formulation, accuracy, storage and computing time requirements are pointed out together with the main aspects of the finite element method for the analysis of the same problem. The large deflection behavior of square plates under a central patch loading (a concentrated load distributed over a finite area) is investigated. Numerical solutions are offered in a general and condensed form for plates with simply supported and clamped boundary conditions, and for a range of patch sizes. The solutions are shown to be of particular value in the assessment of stresses under a patch loading in practical plate problems. The solutions can serve as a useful guide line for the advanced design of the plate components of plated structures.

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