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Large-Amplitude Vibrations of Rectangular Plates

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1 Introduction

The natural frequency of vibration of a thin sheet depends on its geometry and material properties, as well as on the magnitude of the amplitude of vibration. Other conditions remaining unchanged, the larger the amplitude of vibration, the higher is the frequency. Thin plates develop membrane stresses under lateral deflection, which enhance their flexural stiffness. Deflections of the order of plate thickness for thin sheets such as load bearing skins or glass panels are not uncommon [1]. It is established that amplitudes of this magnitude significantly affect the vibration behavior and the resonance characteristics of thin plates [2–4].

Large-amplitude flexural vibrations of rectangular plates for small strains are described by two coupled fourth-order partial differential equations, which were first derived by Herrmann [4] as an extension of the static von Karman [5] equations. The exact solution to these equations is yet unknown. References to other earlier contributions on this subject may be found in [6, 7].

Herein, a general approximate approach is presented for the large-amplitude vibrations of plates having different boundary conditions. The large-amplitude deflection of plates is governed by the large deflection equations of plates [5, 8], which yield the deflected profile and the resulting membrane stresses. Herein, the membrane stresses evaluated are considered as constant throughout the vibration. Subsequently, the frequency is calculated using Herrmann's equations with the constant membrane stresses obtained from the large deflection analysis. An iteration procedure used in obtaining the frequencies allows the plate to take its preferred mode of vibration at the preselected large amplitude. The membrane stresses developed in vibration oscillate between zero (neutral position) and maximum (max deflection), when the plate attains its greatest stiffenss. In the present treatment, frequencies are evaluated with membrane stresses frozen at their greatest values, yielding a constant and maximum membrane stiffness during vibration. The solutions obtained are upper bounds to the true frequency.

The main features of the present analysis are: (*i*) a great reduction in computational effort: (*ii*) obtaining upper bounds to the large-amplitude vibrations of rectangular plates, which together with the lower bound solutions available through small deflection analyses, provide a comprehensive reference basis: (*iii*) using a general numerical method, all the boundary conditions can be dealt with. The method can be extended to cover orthotropic plates: (*iv*) the frequencies obtained are in close agreement with the limited available solutions: and (*v*) the analysis does not allow for shear deformation and the inplane inertia forces.

To demonstrate the potential of the treatment presented, solutions are obtained for simply supported and clamped rectangular plates having different boundary conditions. The results obtained are compared with the available solutions.

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2 Theory

2.1 Governing Equations. Assume a large deflection rectangular plate in static equilibrium under the action of an applied transverse pressure q_{xy}. Let the static deflections be w_s. If the plate is now subjected to a sinusoidally varying exitation force of small magnitude relative to q_{xy}, it would vibrate about the equilibrium position with an amplitude w_d, which would not, by assumption, be large enough the affect the general state of membrane stresses in the plate. The analysis determines the natural frequencies of this plate, for which the membrane stresses remain constant at values relating to w_s. The static deflection, and the membrane forces are expressed by:

$$\left\{\begin{aligned} & w_{s,xxxx}+2w_{s,xxyy}+w_{s,yyyy}-\frac{1}{D}[f_{,yy}w_{s,xx}-2f_{,xy}w_{s,xy} \\ & \qquad +f_{,xx}w_{s,yy}]=q_{xy/D} \\ & f_{,xxxx}+2f_{,xxyy}+f_{,yyyy}=Eh[(w_{s,xy})^2-w_{s,xx}w_{s,yy}] \end{aligned}\right. \tag{1}$$

where a comma followed by subscripts represents partial differentiation with respect to the subscripts. The membrane forces are derived from the following:

$$N_x=f_{,yy};\quad N_y=f_{,xx};\quad N_{xy}=-f_{,xy} \tag{2}$$

The inertia force considered is due to transverse motion and has an intensity of $mw_{d,tt}$. The equation of motion of the plate can now be written by adding $(mw_{d,tt})$ to the flexural equation of the set of equations (1), and interpreting the total deflection w as the sum of $(w_s + w_d)$. The second equation, however, is based on the static deflections w_s for the evaluation of the governing membrane stresses.

$$\left\{\begin{aligned} & w_{,xxxx}+2w_{,xxyy}+w_{,yyyy}-\frac{1}{D}[f_{,yy}w_{,xx}-2f_{,xy}w_{,xy} \\ & \qquad +f_{,xx}w_{,yy}]=\frac{q_{xy}}{D}-\frac{m}{D}w_{d,tt} \\ & f_{,xxxx}+2f_{,xxyy}+f_{,yyyy}=Eh[(w_{s,xy})^2-w_{s,xx}w_{s,yy}] \end{aligned}\right. \tag{3}$$

Subtracting the static component of deflection associated with q_{xy} from the first equation gives:

$$w_{d,xxxx}+2w_{d,xxyy}+w_{d,yyyy}-\frac{1}{D}[f_{,yy}w_{d,xx}-2f_{,xy}w_{d,xy} +f_{,xx}w_{d,yy}]=-\frac{m}{D}w_{d,tt} \tag{4}$$

Assume a simple harmonic motion for the first mode.

$$w_d(x,y,t)=w_a(x,y)\sin(\omega t+\alpha) \tag{5}$$

Substituting equation (5) in equation (4) and cancelling the term sin $(\omega t + \alpha)$ results in the governing frequency equation:

$$w_{a,xxxx}+2w_{a,xxyy}+w_{a,yyyy}-\frac{1}{D}[f_{,yy}w_{a,xx}-2f_{,xy}w_{a,xy}+f_{,xx}w_{a,yy}] =m\omega^2 w_a \tag{6}$$

Equation (6) represents the natural vibrations of the plate with large amplitudes in terms of w and f. For any preselected large amplitude w_a, values of f are first determined from the set of equations (1) relating to the stationary position of maximum deflection, for which $w_s = w_a$. Substitution in equation (6) for the calculated values of f, yields a standard eigenvalue problem.

2.2 Boundary Conditions. For edges parallel to y-axis, (a) flexural boundary conditions, (i) for rotationally free edges $w = 0$ and $w_{,xx} = 0$, (ii) for rotationally fixed edges $w = 0$ and $w_{,x} = 0$; (b) membrane boundary condition, (i) for all the cases treated $f_{,xy} = 0$, (ii) for the inplane condition perpendicular to the edges, two cases are considered, i.e., $f_{,yy} = 0$ to match the simply supported cases, or inplane displacement equated to zero to match the fixed condition.

2.3 Numerical Solutions. First equations (1) are solved. Subsequently, with known values of f, the eignevalue problem given by equation (6) is treated, using finite differences with a mesh having 300–400 nodes over the entire plate. The algebraic eigenvalue problem derived from equation (7) is given by:

$$[A]\{w_a\} = \Omega\{w_a\} \tag{7}$$

where $[A]$ is a square matrix depending on f and representing the left-hand side of equation (6) expressed in finite differences. $\{w_a\}$ is the unknown column eigenvector. Ω is nondimensionalized frequency. Stress function f is found from equations (1) and is used to form $[A]$. The smallest eigenvalue of $[A]$ is found using the power method with the following iteration:

$$\left\{ \begin{array}{l} \{\bar{w}\}_{i+1} = [A]^{-1}\{w\}_i \\ \{w\}_{i+1} = C_i\{\bar{w}\}_{i+1} \end{array} \right. \tag{8}$$

where the subscripts refer to the number of iteration. $\{\bar{w}\}_{i+1}$ is the assumed eigenvector for the $(i+1)$th iteration. The iteration is carried through for $i = 0, 1, \ldots$ until the evaluated eigenvector $\{w\}_{i+1}$ from $(i+1)$th iteration is sufficiently close to the eigenvector $\{w\}_i$ from the preceding iteration. $\{w\}_i$ is then the required eigenvector. C_i is the normalizing factor and can be shown to converge the smallest eigenvalue Ω_1 as $\{w\}$ converges.

3 Discussion of Results

The applicability of the method is demonstrated through treatment of two sets of plates with different flexural and membrane boundary conditions. The solutions presented extend the range of published amplitudes. A third set of solutions for rectangular plates serves to demonstrate the influence of aspect ratios on large-amplitude vibrations. Figures 1 and 2 show the large-amplitude frequencies of square plates with stress free and immovable boundary conditions, respectively. The results from the present analysis yield higher frequencies compared to the available solutions. However, the agreement is close for most practical purposes. Figure 3 represents large-amplitude natural frequencies of simply supported and rotationally fixed rectangular plates of different aspect ratios. The large-amplitude frequency for any given amplitude may be evaluated by simply dividing the given small-amplitude frequency (Ω_s) through the measured ordinate of the curve at the selected amplitude. Note that the solutions presented apply equally to the small vibrations of plates undergoing large deflections under a uniformly distributed loading to a static deflection equal to the large amplitude given herein.

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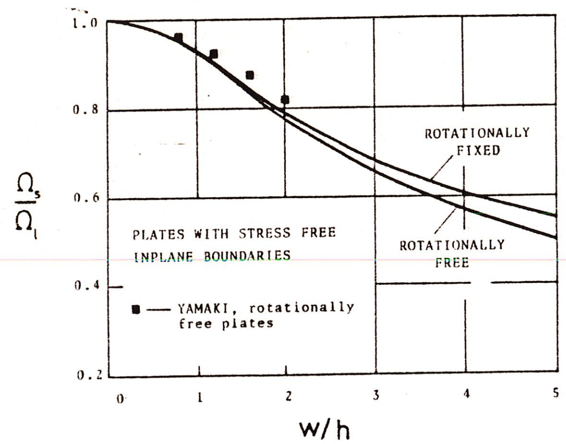
Fig. 1 Large-amplitude frequencies (Ω_l) of square plates with stress-free inplane boundary conditions (Yamaki [3]). Small-amplitude frequencies (Ω_s) are for rotationally free plates 19.68, and for rotationally fixed plates 35.86.



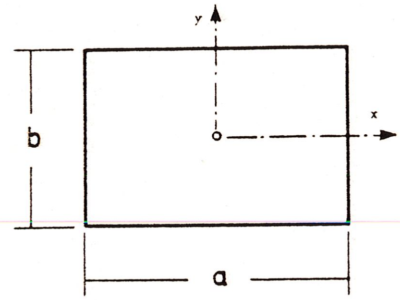
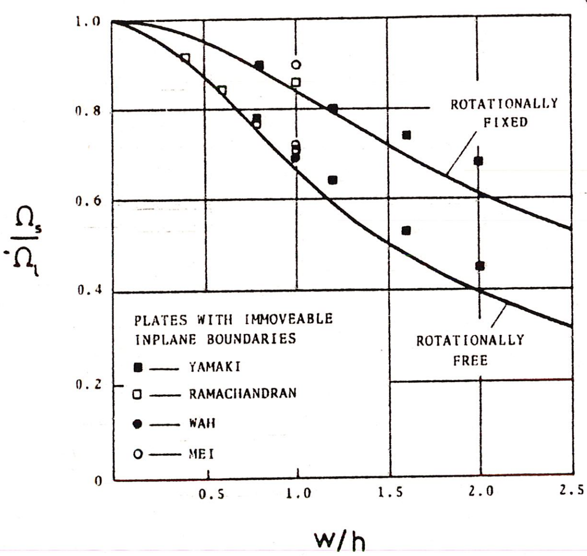


Fig. 2 Large-amplitude frequencies (Ω_l) of square plates with immovable boundaries (Yamaki [3]), Ramachandran [9], Wah [10], Mei [11]. Small amplitude frequencies (Ω_s) are for rotationally free plates 19.68, and for rotationally fixed plates 35.86.



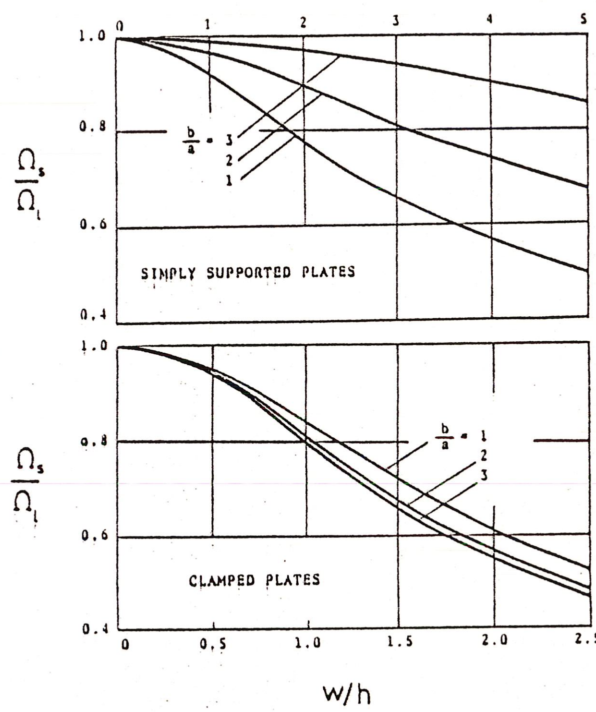


Fig. 3 Large-amplitude frequencies (Ω_l) of rectangular plates with different aspect ratios. Upper section for rotationally free plates with zero inplane normal and shearing stresses, with small-amplitude frequencies (Ω_s) equal to: for $b/a = 1$, 19.68; $b/a = 2$, 12.28; $b/a = 3$, 10.91. Lower section for rotationally fixed plates with immovable boundaries with Ω_s equal to: for $b/a = 1$, 35.86; $b/a = 2$, 24.42; $b/a = 3$, 23.05.