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Moment-Rotation Relation Between Column and Slab

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A method is presented for obtaining lower and upper limits to the rotational stiffness of flat floor plates at a column-slab junction. The method is based on the elastic theory of isotropic thin plates and accounts for the column cross sectional configuration at the column-slab interface, as well as the floor plate's boundary conditions. The flexural stiffness of the slab at a column-slab junction is represented by a moment-rotation coefficient as used in beam and frame analysis. Moment-rotation coefficients for several cases of square plates on central square columns are evaluated and given. The distribution of moment in the slab due to an applied moment at the column-slab junction is studied. A numerical example is included to show the application of the proposed method.

Keywords: bending moments; columns (supports); concrete slabs; **flat concrete plates;** frames; plates (structural members); **reinforced concrete;** rotation; **stiffness; structural analysis;** theory of elasticity.

■ THE APPLICATION OF beamless flat plate floor construction is of special interest due to its aesthetic features or economical considerations in reduced floor heights. In flat plate construction where the floor may be subject to unbalanced gravity loads, wind effects, earthquake forces, the designer is faced with determining the (1) moment which is transferred to the plate, and (2) stresses set up in the floor due to the transferred moment.

In two recent papers,[1,2] Mast reviewed the latest developments in the above field, and in dealing with the second part of the problem proposed a comprehensive analytical method for the evaluation of stresses in the vicinity of columns in flat plates. The analysis conducted by Mast is based on the elastic theory of isotropic plates. The resolution of an applied moment at the column-slab junction into its components of flexure, shear, and torsion is tabulated for a number of peripheral sections around the column. The solutions are shown to agree well with test results.

This paper makes use of the same elastic theory of plates, and in dealing with the first part of the problem describes a simple and useful method for evaluating upper and lower limits to the rotational stiffness of a floor plate at a column-slab junction. The rotational stiffness coefficient may be used to determine the moment transferred to a slab making use of the assumption that an unbalanced moment at a junction is shared between the column and slab due to their respective moment-rotation stiffnesses.

In many flat plate structures, under gravity load, due to symmetry or simplifying assumptions the column moment may be evaluated without using a moment-rotation relation for the floor plate. Where unsymmetrical conditions prevail, or the simplification may not be permissible, the analysis is normally carried out by assuming a certain width of the plate to act as a beam strip with the column and treating the structure as a rigid frame. The effective width of the beam strip for a frame analysis is usually based on engineering judgment. However, this analysis yields lower and upper limits to the effective width of the floor slab which may be considered to act as a beam strip in a frame analysis. By comparing the behavior of the plate with the behavior of the beam strip, further insight into the problem is given which serves as a guide to the designer.

NATURE OF THE PROBLEM

At the column-slab junction, the behavior of both the column and slab are complex. Because

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the concrete floor gains an increase in its effective depth within the column periphery at the column-slab interface, it acts stiffer. The factors which influence the behavior of the junction and floor slab in its vicinity are the material properties, initiation and development of cracks, geometry of the column and slab, and arrangement of the reinforcement.

For the purpose of this paper, the following assumptions are made in line with Reference 1. Concrete behaves as an isotropic material; creep and cracks are not accounted for, noting that the results will be accurate only in the elastic range and must be used with caution in an ultimate behavior analysis. The elastic theory of thin plates[3] (floor represented by its midplane behavior) may be used to represent the floor behavior accurately enough, but the column geometry at the column-slab interface needs to be considered. In this context the rotational stiffness of a slab floor at a column-slab interface is within the two upper and lower limits as defined below.

Lower limit [Fig. 1(i,ii,iii)]

In the lower limit the column is assumed to add no additional bending stiffness to the floor plate at the column-slab interface [Fig. 1(i)]. As a result the floor plate may be assumed to be uniform and continuous at the column-slab interface, having the same thickness and flexural stiffness of the unsupported region. The deformation of the column-slab interface will, therefore, be a continuation of the deformation surface of the plate of the unsupported region. This implies that the governing equation of plate behavior for regions outside the column-slab interface are equally valid for the behavior within the column-slab interface (with appropriate loading, see the Appendix). There is no need, then, for definition and use of any boundary conditions at the column periphery.

In this case the moment transfer between the column and floor plate at the interface takes place through bending stresses in the column and the

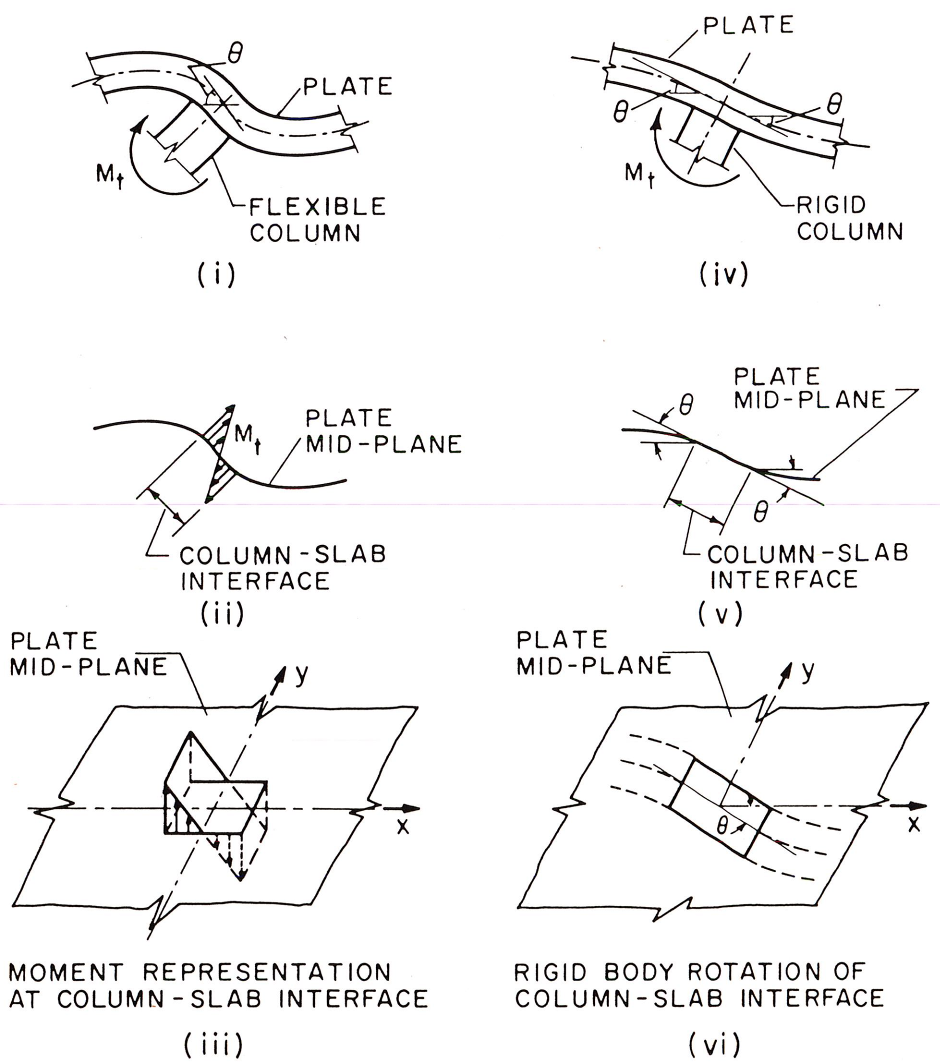

Fig. 1—Column-slab junctions; (i) column exerting no restraint to slab bending; (ii) and (iii) assumed force distribution at column-slab interface for lower limit case; (iv) column exerting infinite restraint to slab bending at interface; (v) and (vi) assumed rigid body rotation of interface for upper limit case

corresponding reacting distributed normal forces on the plate surface. For this analysis the bending moment at the column-slab interface is represented by two linearly varying distributions of pressure in the form of two triangles (positive and negative) as shown in Fig. 1(ii) and 1(iii). The assumed distribution is equivalent to the total applied load M_t with zero total transverse force on the plate. The resulting rotation θ is defined as the rotation of the plate in the direction of the applied moment at the center of column-slab interface.

Upper limit [Fig. 1(iv,v,vi)]

An upper limit to the moment-rotation stiffness of the plate may be obtained. This is done by assuming that the column makes the floor plate infinitely stiff in bending within the column-slab interface such that an applied moment M_t at the column-slab junction will result in a rigid body rotation of the interface accompanied by the elastic deformation of the plate in the region outside the interface. The plane of the interface remains undeformed and rotates through an angle determined by the elastic rotational stiffness of the plate. If the interface is given a unit rigid body rotation, the elastic plate equation outside the column periphery can be solved using the same governing equation as in the previous case. However, at the periphery, due to the imposed rigid body rotation, appropriate boundary conditions need to be specified (plate displacement same as periphery, plate slope same as column-slab interface, see Appendix). From the solutions obtained and considering equilibrium of forces on the outer boundaries of an arbitrary peripheral section of the plate [such as the section shown in Fig. 1(vi) or Fig. 2 of Reference 1], the applied moment M_t corresponding to the imposed rotation can be evaluated directly.

In practical situations the rotational stiffness of the floor plate at the column-slab interface lies between the two extremes outlined above. For a given column cross section thinner plates are influenced more by the stiffening of the column and their stiffness approaches the upper limit as the floor thickness is reduced.

EVALUATION OF COEFFICIENTS

The plate equations may be solved using any of the standard numerical procedures, noting the moment value in the lower limit case and the boundary conditions in the upper limit case. In this paper solutions were obtained by using the finite difference method with a graded mesh.[4] Making use of symmetry and antisymmetry, the first quadrant of the plate is subdivided into a fine orthogonal mesh, which is graded to be finer over the column slab-interface (see Fig. 3). The plate equations together with the boundary conditions are written for the nodes, and are arrayed into a set of matrix equations which are solved using a computer. (Details of the analysis are given in the Appendix).

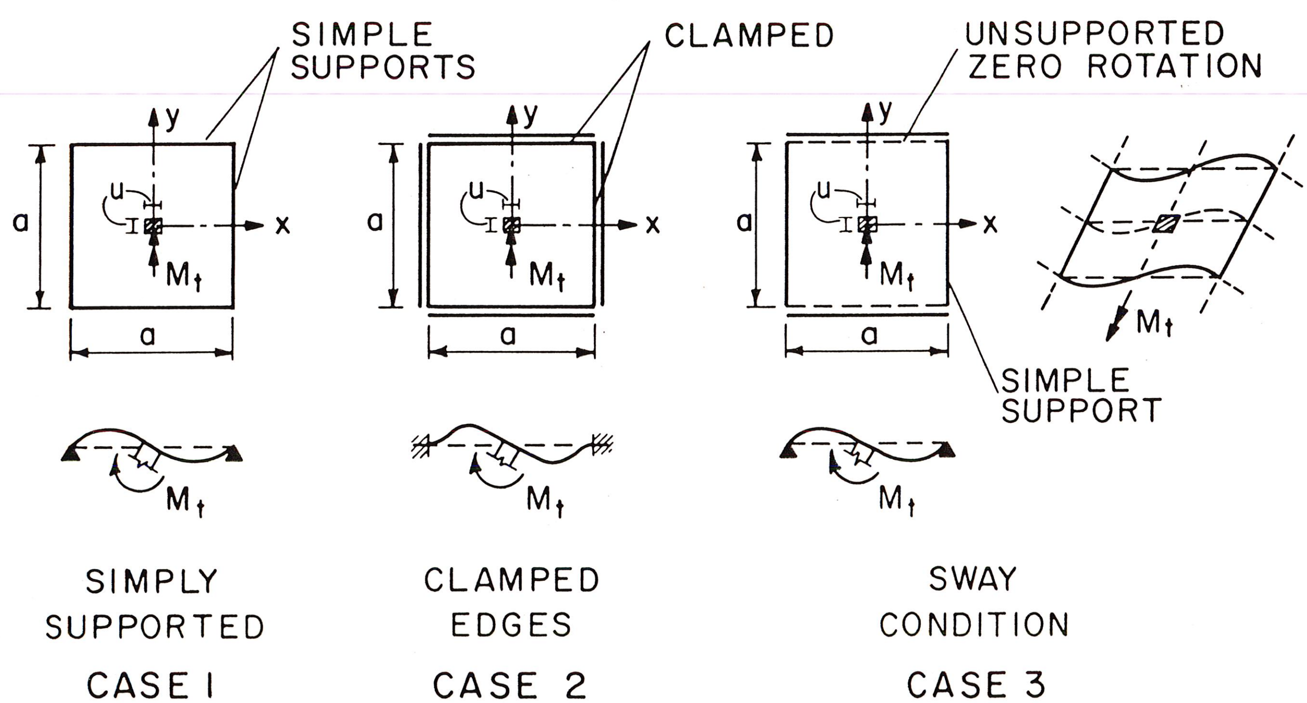

Fig. 2—Square plates on central columns; dimensions and boundary conditions of three cases analyzed

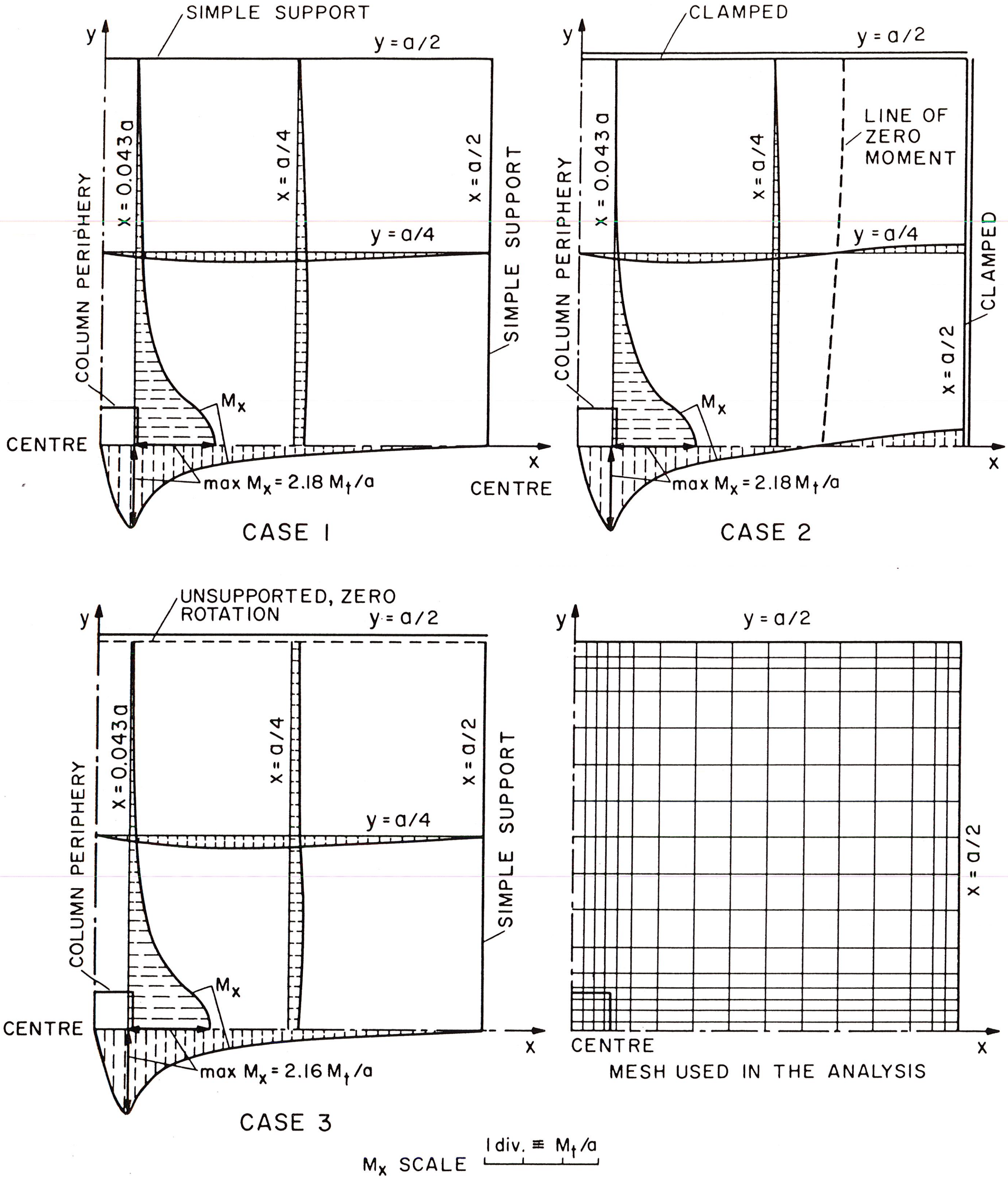

Fig. 3—Square plate on central column ($u/a = 0.1$); distribution M_x in the first quadrant of floor slab due to applied moment M_t (in x direction) at column-slab junction; and plate subdivision used for analysis

SQUARE PLATE ON CENTRAL COLUMN

Square panels on a central square column are important because they simulate interior panels of a building. The following three standard cases (shown in Fig. 2) are analyzed for the lower limit conditions.

Case 1—Simply supported, in which all edges are resting on rigid supports, but are free to rotate [see Eq. (A2), (A5), and (A6) in Appendix].

Case 2—Clamped conditions, in which all edges are resting on rigid supports, but are rotationally fixed [see Eq. (A2), (A7), and (A8) in Appendix].

Case 3—Sway condition, which applies to a typical unit of an interior panel of a building under lateral load. From the geometry of the deformation of the swayed panel, it may be seen that (Fig. 2, Case 3) the boundary conditions perpendicular to the direction of sway are equivalent to those of a simply supported boundary; the other two boundaries deflect freely with zero rotation across the boundary [see Eq. (A4) and (A8) in Appendix].

In each case solutions are obtained for square columns, for three different ratios of column side u to plate side a equal to $u/a = 0.05$, 0.1 and 0.15.

The moment-rotation coefficients ($k_1 = M_t/D\theta$, where $D = EI$ for zero Poisson's ratio) are listed in Table 1. In each case an equivalent beam strip moment-rotation stiffness is evaluated and given in the same table for comparison. The equivalent beam strip moment-rotation coefficient k_1 is based on the rotational stiffness shown by a beam of width equal to the total width of the plate and boundary conditions as shown immediately below each case in Fig. 2, and being subject to a moment M_t distributed uniformly across the width. [For example, in Case 1 $M_t = 2a(3EI\theta)/(a/2)$, hence $k_1 = 12$.]

As can be seen from Table 1, plate solutions give rotational stiffnesses for the three cases which are close to one another for the same column area. This indicates that the rotational stiffness of a plate is primarily influenced by the dimensions of the column and is not sensitive to the outer boundaries of the plate, especially for smaller column sizes. Further, if the columns are small enough (of the order considered here $u/a < 0.15$), the span of the plate in either direction may be considered to have little effect on the moment-rotation relation. It is well known that the behavior of the central region of Case 2 plate (clamped) may be associated with that of Case 1 plate (simply supported) having a shorter effective span.

The distribution of moment M_x in the first quadrant of the plates, due to an applied moment M_t at the column-slab interface is shown in Fig. 3 for the three cases with column $u/a = 0.1$. The distributions are anti-symmetrical in Quadrants 2 and 3 and symmetrical in Quadrant 4. Maximum moments occur in the direction of the applied moment along the x-x centerline of the plate at the column periphery as shown in Fig. 3. The maximum values of bending moments obtained are summarized in Table 1.

The analysis shows that the distribution of bending moment in the vicinity of the column is not affected by the plate's outer boundary conditions and that the maximum values obtained for the same column areas of the three cases are within 3 percent of one another. It should, however, be noted that, from the lower limit assumptions made for the rotational stiffness, the moments as shown in Fig. 3 are concentrated at the column center line with a pronounced peak. The upper limit solution would result in the same moment spreading over a wider range giving a smaller peak in M_x. The bending moments given are therefore likely to exceed the true values. Their relative values and distributions provide a qualitative insight into the plate behavior. For the design value of bending

TABLE 1—SQUARE PLATE ON CENTRAL COLUMN, SUBJECTED TO MOMENT M_t (IN X DIRECTION) AT COLUMN-SLAB INTERFACE*

Case	Column size u/a	Moment-rotation coefficient k_1: Present solution	Moment-rotation coefficient k_1: Equivalent beam	Maximum M_x coefficient k_2
Case 1 Simply supported	0.05	4.17	12	4.54
	0.1	5.53	12	2.18
	0.15	6.78	12	1.40
Case 2 Clamped	0.05	4.25	16	4.54
	0.1	6.24	16	2.18
	0.15	7.87	16	1.41
Case 3 Sway condition	0.05	4.10	12	4.53
	0.1	5.40	12	2.16
	0.15	6.59	12	1.37

*Moment-rotation coefficients ($k_1 = M_t/D\theta$) of floor plate, and maximum values of moment ($M_x = k_2 M_t/a$) in floor plate for varying column dimensions (u/a).

moments, which is the second part of the problem as outlined in the introduction, reference should be made to the formulas given in Table 1 of Reference 1.

DESIGN EXAMPLE

To illustrate the proposed method, consider a unit of a typical square floor slab (120 x 120 in.) around a square column (6 x 6 in.). The column is 100 in. long and is fixed at the base (Fig. 4). Let the bending moment acting at the column-slab junction be 100 in.-kips as shown in Fig. 4. It is required to determine the magnitude of the moment which is transferred to the floor plate.

For the column; the rotation at the top is:

$$\theta_c = \frac{M_c h}{4EI} \qquad (1)$$

where

M_c = applied moment at column
h = height of column

For the slab, the rotation at the junction is:

$$\theta = \frac{M_t}{k_1 D} \qquad (2)$$

where

M_t = total applied moment
k_1 = rotational stiffness coefficient at column-slab interface

D = flexural rigidity of slab

For equal rotations:

$$\frac{M_c h}{4EI} = \frac{M_t}{k_1 D} \qquad (3)$$

From equilibrium:

$$M_c + M_t = 100 \qquad (4)$$

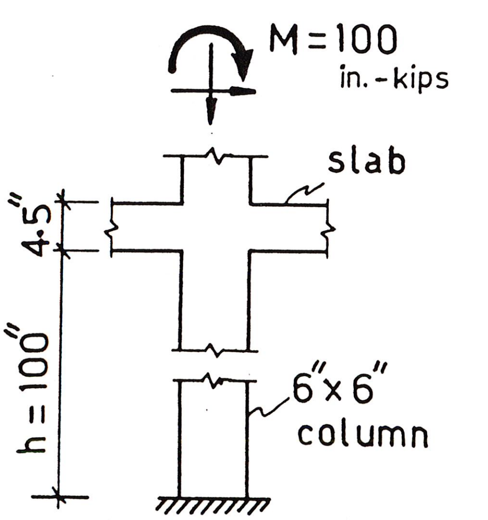

Fig. 4—Design example; column and floor slab

Using the two relations [Eq. (3) and (4)], M_t and M_c can be evaluated directly. For $u/a = 6/120 = 0.05$, the coefficient k_1 varies from 4.10 to 4.25 depending on the boundary conditions of the floor slab (Table 1). For the sway condition, $k_1 = 4.10$.

On substitution, Eq. (3) becomes:

$$\frac{M_c \times 100}{4EI \times 6^4/12} = \frac{M_t}{4.10E \times 4.5^3/12}$$

for $\nu = 0$.

Hence, the moment transferred to the slab M_t = 88 in.-kips.

For the stresses set up in the slab, Reference 1 or similar works should be consulted. But from the present analysis, the maximum intensity of moment at the column periphery is given from the approximate coefficient k_2 of Table 1 as:

$$M_x = \frac{4.53 \times 87}{120} = 3.3 \text{ in.-kips per in.}$$

CONCLUSIONS

An analysis, based on the elastic theory of plates, is presented for obtaining lower and upper limits for the rotational stiffness of flat floor plates at a column-slab junction. Based on the analysis three cases of square plates on central columns with several column dimensions and floor plate's outer boundary conditions are discussed. The moment-rotation stiffnesses obtained are tabulated and are compared with the values given by simple beam strip theory.

The analysis has shown that the moment-rotation stiffness exhibited by a floor plate depends largely on the column dimensions and is not significantly affected by the span of the floor plate and its outer boundary conditions. This implies particular caution in the use of the beam strip approach, where the rotational stiffness is a function of plate span and boundary conditions.

It is shown that for plates on a central column, the magnitude and distribution in the vicinity of the column of an applied moment at the column-slab junction is not greatly influenced by the outer boundary conditions of the floor plate if column dimensions are small enough ($u/a < 0.15$), but depends on the column dimensions.

ACKNOWLEDGMENT

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APPENDIX

Notation

a	=	plate floor side, in. (cm)
D	=	flexural rigidity of slab per unit length, $Et^3/12(1-\nu^2)$, in.-lb (kgf-cm)
E	=	Young's modulus of elasticity, psi (kgf/cm^2)
h	=	height of column, in. (cm)
I	=	Moment of inertia, in.^{4} (cm^4)
k_1	=	$M_t/D\theta$ rotational stiffness coefficient at column-slab interface
k_2	=	$M_x/(M_t/a)$ ratio of maximum moment to average applied moment
M_c	=	applied moment at column, in.-kips (m-kgf)
M_t	=	total applied moment to slab at column-slab interface, in.-kips (m-kgf)
M_x, M_y	=	bending moments in x and y directions respectively, in.-kips per in. (m-kgf/m)
q	=	intensity of transverse pressure, psi (kgf/cm^2)
t	=	plate thickness, in. (cm)
u	=	column dimensions in x and y directions, in. (cm)
w	=	transverse deflection of slab, in. (cm)
θ	=	rotation of slab in x direction at center of column-slab interface
θ_c	=	rotation at top of column
ν	=	Poisson's ratio

Metric equivalents

Multiply	By	To obtain
in.	0.0254	m
in.-kips	11.521	m-kgf
in.-kips per in.	453.59	m-kgf/m

Theoretical background

The plate bending is expressed in the small deflection range by the following partial differential equation:[3]

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = q/D \qquad \text{(A1)}$$

where the subscripts after the comma each imply differentiation of the function with respect to that variable.

The boundary conditions at the four edges are:

1. *For rigidly supported edges*

$$w = 0 \qquad \text{(A2)}$$

or for unsupported edges

$$\text{At } x = \pm a/2 \qquad w_{,xxx} + (2-\nu)w_{,xyy} = 0 \qquad \text{(A3)}$$

$$\text{At } y = \pm a/2 \qquad w_{,yyy} + (2-\nu)\,w_{,xxy} = 0 \qquad \text{(A4)}$$

2. *For rotationally free edges*

$$\text{At } x = \pm a/2 \qquad w_{,xx} + \nu w_{,yy} = 0 \qquad \text{(A5)}$$

$$\text{At } y = \pm a/2 \qquad w_{,yy} + \nu w_{,xx} = 0 \qquad \text{(A6)}$$

3. *For rotationally fixed edges*

$$\text{At } x = \pm a/2 \qquad w_{,x} = 0 \qquad \text{(A7)}$$

$$\text{At } y = \pm a/2 \qquad w_{,y} = 0 \qquad \text{(A8)}$$

At the column-slab interface, for the lower limit case Eq. (A1) is valid within the interface region with the applied transverse pressure q to be specified as outlined earlier. For the upper limit case, Eq. (A1) applies to the region outside the column periphery and needs to be solved over this domain only. From Fig. 1 (vi) the conditions due to a unit rotation at the periphery for edges parallel to the y axis are:

$$w = 1 \times \frac{u}{2} \text{ and } w_{,x} = 1 \qquad \text{(A9)}$$

and at the periphery edges parallel to the x axis:

$$w = 1 \times x \quad \text{and } w_{,y} = 0 \qquad \text{(A10)}$$

From the solution obtained, for any peripheral section as shown in Fig. 1(vi) with boundaries at $x = \pm Ua$ and $y = \pm Va$, the equilibrium of forces as shown in Fig. 3 of Reference 1 gives the value of the associated applied moment M_t as follows (Reference 1, Table 3):

$$M_t = 4 \int_o^{Va} M_x dy + 4 \int_o^{Ua} M_{yx} dx + 4Ua \int_o^{Va} Q_x dy + 4 \int_o^{Ua} q_y x dx \qquad \text{(A11)}$$

For obtaining a solution, the plate is subdivided into a fine mesh. Plate Eq. (A1) together with the boundary conditions are expressed at the nodes in terms of finite difference approximations developed for a graded mesh.[4] The difference equations obtained are arrayed in the following matrix form:

$$[A]\,\{w\} = \{q\} \qquad \text{(A12)}$$

where

$[A]$	=	square matrix comprising finite difference coefficients of nodes
$\{w\}$	=	column matrix expressing deflection of slab
$\{q\}$	=	column matrix for transverse pressure which is zero at all nodes, except at column-slab interface, where it is used to express applied moment

Matrix Eq. (A12) is solved for w using a modified partitioning technique and a computer. Bending moments are then evaluated from the known deflections using the following expressions:

$$M_x = -D\,(w_{,xx} + \nu w_{,yy}) \qquad \text{(A13)}$$

$$M_y = -D\,(w_{,yy} + \nu w_{,xx}) \qquad \text{(A14)}$$

For the values of moments quoted in the paper, Poisson's ratio is assumed equal to zero.

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SYNOPSIS: A method is presented for obtaining lower and upper limits to the rotational stiffness of flat floor plates at a column-slab junction. The method is based on the elastic theory of isotropic thin plates and accounts for the column cross-sectional configuration at the column-slab interface, as well as the floor plate's boundary conditions. The flexural stiffness of the slab at a column-slab junction is represented by a moment-rotation coefficient as used in beam and frame analysis. Moment-rotation coefficients for several cases of square plates on central square columns are evaluated and given. The distribution of moment in the slab due to an applied moment at the column-slab junction is studied. A numerical example is included to show the application of the proposed method.

REFERENCE: Aalami, Bijan, "Moment-Rotation Relation Between Column and Slab," ACI JOURNAL, Proceedings V. 69, No. 5, May 1972, pp. 263-269.