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Rotational Stiffness of Concrete Slabs

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Using a finite element elastic plate analysis, lower and upper limits to the rotational stiffness of a flat slab at the column-slab junction are evaluated for a number of typical square slabs on a central column. The slabs represent unit panels of multiple-slab floors. The stiffnesses determined are presented in a graphical form and may be used in the equivalent-frame method in design of multiple square slabs. The results are comprehensive and effectively serve to clarify the picture of slab stiffness behavior. They also provide a useful basis for the accuracy assessment of the current approximate methods of stiffness evaluation. A comparison with the related recommendations of ACI 318-71 shows that the Code underestimates the slab stiffness for certain column-slab geometries. The resulting lower bound solutions for slab stiffness and maximum moment coefficients closely agree with those previously available. A numerical example is given to illustrate the use of the data offered.

Keywords: bending moments; building codes; columns (supports); elastic analysis; finite difference theory; **finite element method**; flat concrete plates; **flat concrete slabs**; flexural strength; frames; reinforced concrete; **stiffness**; structural analysis.

■ FOR MULTIPLE SQUARE OR RECTANGULAR SLABS, reinforced in more than one direction, with or without beams between supports ACI 318-71[1] recommends two design procedures. (1) The *direct design method* which is simple but limited in scope, in that, it applies to panels satisfying certain geometric and loading limitations. (2) The more general *equivalent frame method,* in which the structure is assumed to be made up of equivalent frames, each consisting of a row of equivalent columns and slab-beam strips bounded laterally by centerlines of the panels on each side of the lines of columns or supports. An elastic analysis is suggested with recommendations for the evaluation of slab stiffness to be used in the equivalent-frame system. The suggested equivalent-frame analysis is given more generality in the new Code and may be used for slab systems with or without beams between supports, thus in scope embracing both the two-way slab and flat slab design procedures of the previous Code, ACI 318-63.[2]

A comprehensive and interesting survey and historical report on flat slabs from their introduction in the United States in 1906 up to 1963 is given by Sozen et al.[3] Corley and Jirsa[4] review the recent literature and describe the background to ACI 318-71 on the proposed equivalent-frame method. Further related work on the rotational stiffness of flat slabs are reported in References 5 and 6.

In ACI 318-63 the two-way action of the slab was completely disregarded. ACI 318-71 recognizes the plate behavior of slabs and allows for the two-way action in the equivalent frame method, in which the slab stiffness is modified with due regard to the transverse (normal to direction of equivalent frame) geometry of slab and column. (This will be discussed in the section on comparison with the Code.) Furthermore, to account for the direct moment transfer from one slab to the next in two-way plate action, the Code introduces a torsional element. The stiffness of this torsional member is coupled with that of the column to maintain the unidirectional concept of the equivalent frame method and yet allow for a basically two-way transfer to materialize. The concept adopted in the new Code is sound and intelligent, but due to the paucity of analytical data, the formulas recommended for the evaluation of stiffnesses in the equivalent frame method are rather intuitive in nature and are derived from simplified theory with some experimental support.[4] It is clear that effectiveness and success of the frame analysis rests primarily on the correct evaluation of the equivalent column and slab stiffness coefficients, for which, to date, no general and simple method with reasonable accuracy is proposed. The second author defines upper and lower limits to the rotational stiffness of a slab at column-slab interface; on the basis of which, using plate theory and finite differences, lower bound values are calculated and reported for a number of typical cases (see Reference 7).

Herein, using the elastic theory of plates, as suggested in ACI 318-71 and the finite element method, an attempt is made to fill the gap in the literature on the moment-rotation stiffness of slabs. New data are offered on both the lower and upper limits to the stiffness of slabs. The evaluated stiffnesses are compared with the previously available lower bound cases[7] with which a close agreement is achieved. In the light of the new information obtained, the relating recommendations of ACI 318-71 are reexamined to check the proposed idealization and its accuracy. The influence, on the plate stiffness, of the plate's longitudinal and transverse boundary conditions are studied and their significance clearly demonstrated. A numerical example illustrates the application and scope of the work presented.

ANALYSIS AND STIFFNESS EVALUATION

The moment-rotation of a plate at its column-slab junction can be expressed by the slab stiffness coefficient k_x, based on which the unbalanced moment at the junction is divided between the column and slab. Strictly speaking, the slab stiffness depends on its geometry, boundary conditions and the column (or capital) cross-sectional configuration. It is also influenced by the material properties of the concrete used and the distribution of reinforcing bars. Moreover, for the same panel, keeping all the preceding geometry and material parameters constant, the stiffnesses of the cracked and uncracked slabs would be different. The nonlinear behavior of cracked sections further complicates the problem. The exact evaluation of the stiffness for a specific panel with due consideration to the stated parameters would be too involved and time consuming, and also the results would be too limited for application, referring to the particular cases studied. A rational approach in such a case is the development of general, simple and yet reliable approximate methods for use in everyday design.

Lower and upper limit approach

Herein the lower and upper limit approach to the plate stiffness is adopted and for each panel two limiting values for the plate stiffness are determined. For the lower bound, the column is assumed to impart no additional stiffness to the slab at the column-slab junction. The rotation of the slab is thus the same as that of a uniform and complete plate loaded by a moment distributed over the column-slab interface.[7] For the upper bound solution the slab is assumed infinitely stiff within the column area surrounded by the elastic slab proper, built in along the periphery of the column. An applied moment at the junction results in the rigid body rotation of the column-slab interface accompanied with the elastic deformation of the plate. The actual stiffness of the plate lies between the two limits, very likely close to the upper limit.[8]

Herein the finite element method is employed to obtain the limiting values of slab stiffness. As it is customary, the effects due to cracking and variation in distribution of steel are disregarded. The plate is assumed to be linearly elastic, homogeneous, and isotropic undergoing small deflections. Rectangular plate bending finite elements are used with three degrees of freedom at each node (transverse displacement and two rotations). Details of the finite element formulation, its accuracy and convergence are discussed in Reference 9. Furthermore, a convergence study with respect to rotational stiffness of the slab is made herein and the results are given in the appendix.

Plate symmetry

For the present analysis, advantage is taken from the symmetry of the plate and a 12 by 12 finite element mesh is used for a quarter of the plate. Moreover, to improve accuracy, the grading is made finer within the column-slab interface and near the column face, where the stress gradients are high. For the lower bound cases, the applied moment is represented by a triangularly distributed force over the column-slab interface as outlined in Reference 7. To simulate the triangularly distributed force more accurately in the finite element analysis and consequently achieve accelerated convergence, a consistent load matrix

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presentation is used. For the upper bound cases, the analysis is exactly similar to that of the lower bound except that the thickness of the plate elements within the column-slab junction are taken to be much higher than that of the slab proper to simulate a locally rigid condition.

It should be emphasized that the slab stiffnesses derived herein, account fully for the biaxial bending of the slab as well as its torsional contribution, and therefore correspond to combined effects of bending stiffness of slab K_s and torsional stiffness of the torsional element K_t as defined by ACI 318-71.

Fig. 1 shows a unit square panel with a central square column (or capital) with four different boundary conditions. The applied moment and resulting rotation vectors are designated by double-headed arrows. The boundary conditions selected reveal several extreme limits of a square unit panel in an array of multiple slabs on an orthogonal grid of columns. The most severe case for moment transfer between column and slab is usually the side-sway condition designated in Fig. 1 as SF, in which for a sway in x-direction the boundary conditions of the unit panel are represented exactly. In conjunction with the sway condition a clamped-free (CF) case is also considered to illustrate the influence of longitudinal boundary conditions. The simple-simple (SS) and clamped-clamped (CC) cases supply valuable information on the effects of the outer boundary condition assumptions which may be made for the evaluation of unit panel moments and stiffnesses. For each case the range of column (capital) size c to span length l of c/l = 0.00, 0.05, 0.10, 0.15, and 0.20 is considered.

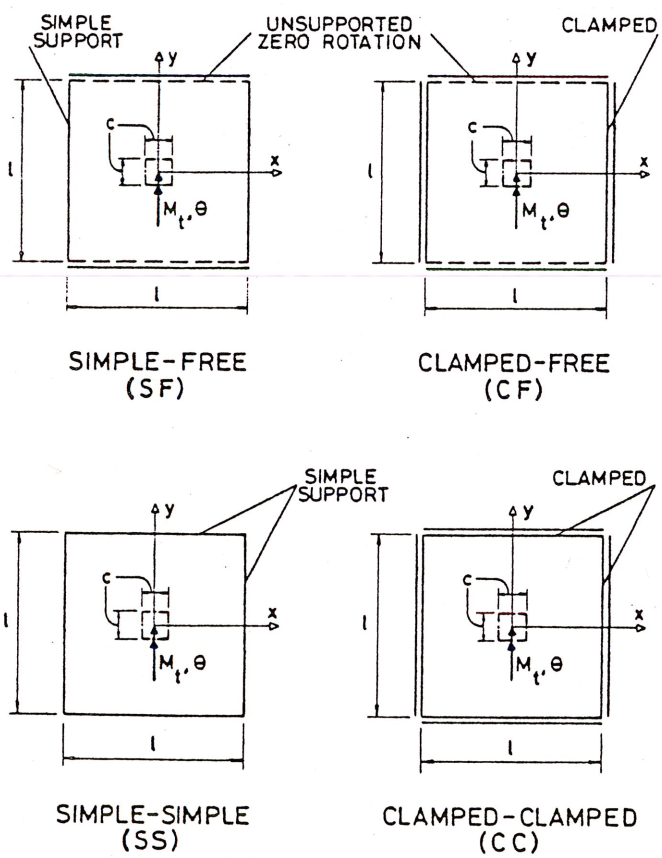

Fig. 1—Unit plate on central column; boundary conditions analyzed

Finite difference solutions

In Table 1 the results of the present analysis are compared with the finite difference solutions of Reference 7 for the lower bound cases available. In comparing the present finite element solutions and the quoted finite difference results it should be noted that the finite difference solutions may be expected to give higher deformations, hence a lower stiffness, contrary to the finite element solution which yields lower deformations and consequently a higher stiffness. The actual lower bound stiffnesses of the panels should be between the corresponding two solutions listed in Table 1. The results, however, are close enough to one another for each to be regarded as a good estimate to the lower bound stiffness.

The maximum lower bound moments given are for zero Poisson's ratio and occur just inside the column face. In the present analysis the maximum moment is obtained by extrapolation of moments at center of plate elements (therefore only two significant figures are given). The moment coefficients k_m are ratios of the maximum slab moment m_x to the value of applied moment over the column width m_t/c. Coefficients k_m from Reference 7 are duly converted to this presentation for comparison. The agreement between the two types of solutions is remarkably good. It should be noted that the nearly equal nondimensional moment co-

TABLE 1 — UNIT PLATE SUBJECTED TO MOMENT AT COLUMN-SLAB INTERFACE. SLAB STIFFNESS AND MAXIMUM MOMENT COEFFICIENTS

Case	Relative column size, c/l	Stiffness coefficient, k, (lower): Present analysis	Stiffness coefficient, k, (lower): Reference 7	Maximum moment coefficient, k_m: Present analysis	Maximum moment coefficient, k_m: Reference 7
Simple-free (SF)	0.05	4.27	4.10	0.21	0.227
	0.10	5.52	5.40	0.20	0.216
	0.15	6.71	6.59	0.20	0.206
Simple-simple (SS)	0.05	4.36	4.17	0.21	0.227
	0.10	5.66	5.53	0.20	0.218
	0.15	6.91	6.78	0.20	0.210
Clamped-clamped (CC)	0.05	4.81	4.25	0.21	0.227
	0.10	6.44	6.24	0.20	0.218
	0.15	8.11	7.87	0.20	0.212

efficients for different c/l ratios imply that, for any given applied moment M_t at column slab junction, smaller columns would have higher intensities of maximum moment (nearly proportional to the inverse of their size c for the range covered).

Variation with column stiffness

Fig. 2 illustrates the variation of both the upper and lower limits of slab stiffness with column (capital) size c/l. The results clearly indicate that for the range of column sizes considered the longitudinal boundary conditions (parallel the x-axis in Fig. 1) have little influence on the slab's rotational stiffness for a longitudinal equivalent frame. This conclusion substantiates the validity of the Code's proposed unidirectional consideration of equivalent frames. The effects of the transverse boundary conditions (parallel to the y-axis in Fig. 1) are, as expected, more pronounced, especially in the upper bound cases and for higher values of c/l.

COMPARISON WITH CODE

The simple-free boundary conditions (SF, Fig. 1) represent a typical middle panel of a multiple-slab floor with square column spacing under lateral loads causing side sway (Fig. 3). This condition, usually provides the most severe unbalanced moment and is compared herein with the recommendations of ACI Codes. Throughout the comparison, Poisson's ratio is taken as zero for concrete.

For the purpose of stiffness evaluation to be used in an equivalent frame analysis, ACI 318-63 considers the plate as a unidirectional beam strip, and recommends a full effective width l in the transverse direction with moments of inertia for column-slab interface and slab as shown in Fig. 4a, from which using beam theory, the slab stiffness coefficient $\bar{k}_x$ can be easily derived to be:

$$\bar{k}_x = 12\left(\frac{1}{1-\frac{c}{l}}\right)^3 \qquad (1)$$

The variations of the preceding $\bar{k}_x$ with different column sizes is shown in Fig. 5 together with the upper bound and lower bound values from the present finite element analysis. ACI 318-63 predicts significantly stiffer panels, which is to be expected, as the infinitely rigid sitffness assumed for the column-slab interface is extended beyond the column periphery in the transverse direction to the boundaries. The stiffnesses thus derived clearly overestimate the slab stiffness to a much higher degree than the upper bound solutions proposed herein.

In an attempt to correct the aforementioned shortcoming, ACI 318-71 improves the slab stiffness calculation by considering the rigid slab region to be limited to within the column periphery. Thus the moments of inertia to be used for the slab and the column-slab interface are modified as shown in Fig. 4b. From information given in

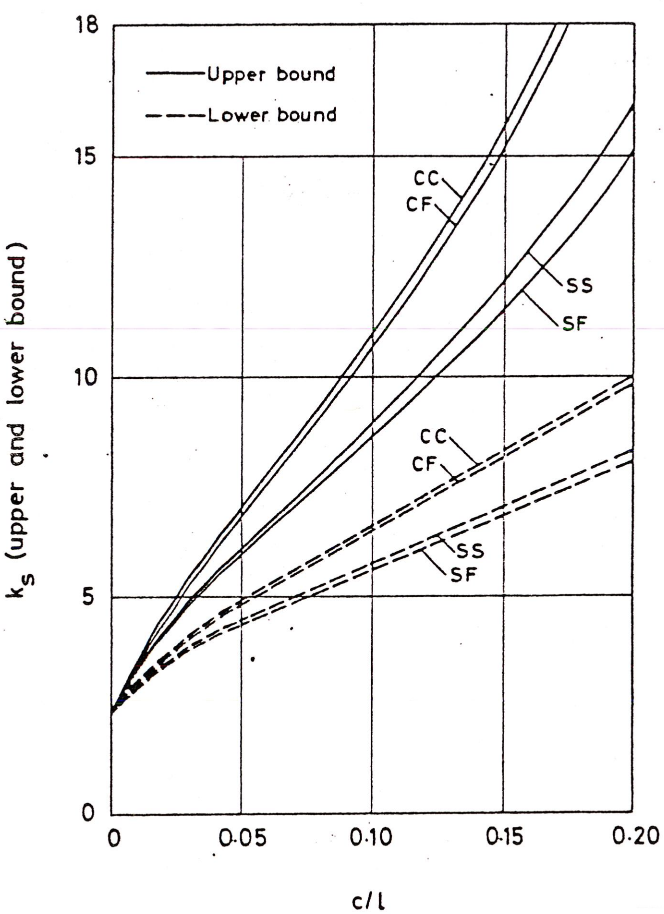

Fig. 2—Lower and upper bound slab stiffness coefficients from present analysis

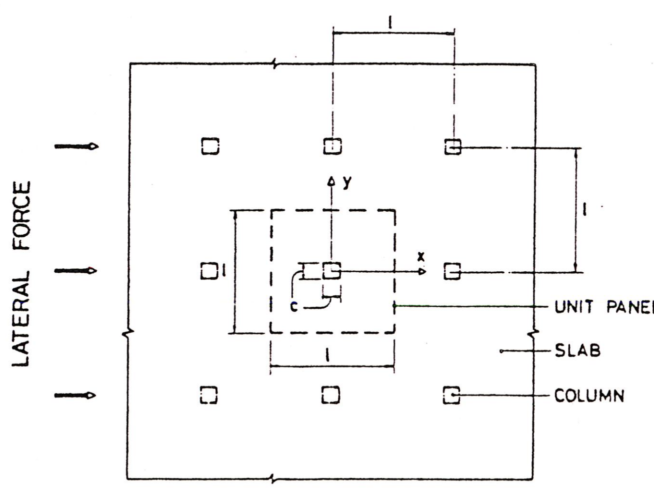

Fig. 3—Typical interior square unit panel under side-sway loading

Fig. 4b and simple beam theory the stiffness coefficient of the slab is:

$$k_s = 4 \div \left[\frac{\left(1-\frac{c}{l}\right)^3}{3} + \left(1-\frac{c}{l}\right)^3 \frac{c}{l} \left(1-\frac{c}{2l}\right) + \frac{\left(\frac{c}{l}\right)^2}{2} \left(1-\frac{c}{l}\right)^2 \left(1-\frac{c}{3l}\right) \right] \quad (2)$$

which is plotted on Fig. 5. Note that for $c/l = 0$, beam strips of Fig. 4a and 4b reduce to a beam of constant moment of inertia $I = lh3/12$ for which $\theta = M_t l/(12EI)$ and therefore $k = M_t/(D\theta) = 12$. Moreover, ACI 318-71 introduces a torsional strip in the transverse direction with torsional stiffness K_t. This is assumed to accommodate for moment transfer between the adjacent panels and between column and slab. Thus the stiffness of the slab and the torsional strip as defined by the Code, may be combined to represent the stiffness of the entire slab. The equivalent slab stiffness K_{es} so obtained may rightly be compared with the slab stiffness derived from two-dimensional plate solution offered herein. The equivalent slab stiffness K_{es} may be evaluated from the following expression:

$$\frac{1}{K_{es}} = \frac{1}{K_s} + \frac{1}{K_t} \quad (3)$$

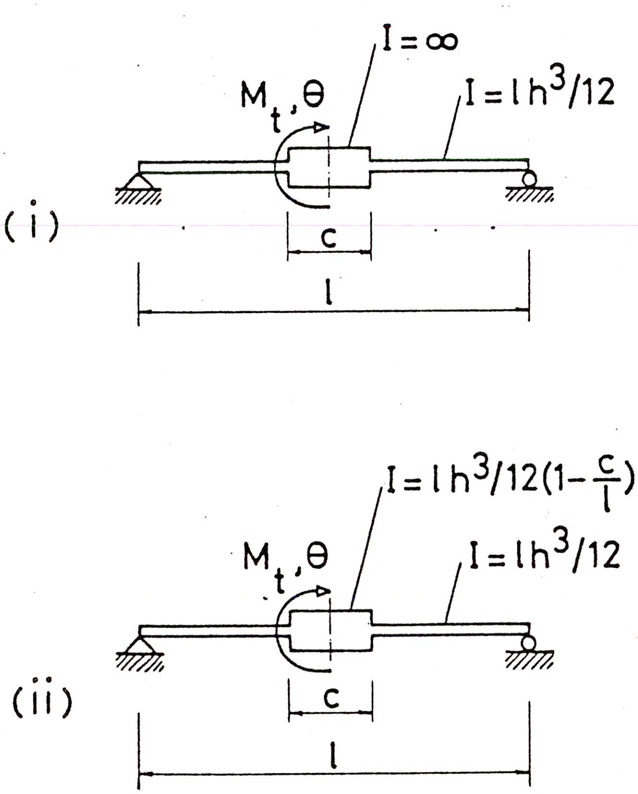

Fig. 4—Unit panel beam strip idealization for slab stiffness calculation as recommended by (a) ACI 318-63 (b) ACI 318-71

or in terms of stiffness coefficients

$$\frac{1}{k_{es}} = \frac{1}{k_s} + \frac{1}{k_t} \quad (4)$$

The preceding equation simply states that the flexibility of the system is the sum of the flexibilities of the individual members connected in series. This is, of course, analogous to the equivalent column stiffness introduced in ACI 318-71. The value of K_t in this case, as proposed by the code, is

$$K_t = 2\frac{9E}{l(1-c/l)^3}\left(1-0.63\frac{h}{c}\right)\frac{h^3c}{3} \quad (5)$$

from which

$$k_t = \frac{K_t}{\frac{Eh^3}{12}} = \frac{72c/l}{\left(1-\frac{c}{l}\right)^3}\left(1-\frac{0.63}{\frac{c}{l}\left(\frac{l}{h}\right)}\right) \quad (6)$$

Substitution of the values for k_s and k_t from Eq. (2) and (6) into Eq. (4) results in the value of k_{es} which for typical values of $l/h = 30$ and 20 is plotted in Fig. 5. Fig. 5 illustrates that the equivalent stiffness coefficient k_{es}, which is ob-

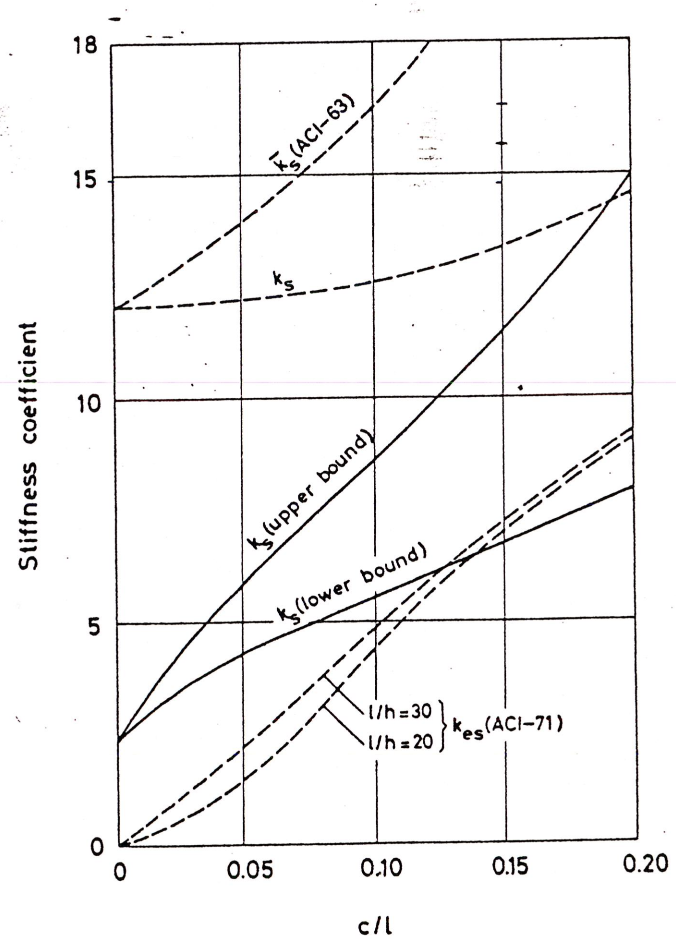

Fig. 5—Slab stiffness coefficients; comparison between ACI Codes and the present elastic plate solutions

tained from the stiffness coefficients of the individual elements as proposed by ACI 318-71 underestimates the elastic stiffness which is expected to be close to the upper bound solution. For higher values of c/l, which applies to flat slabs with capital, the discrepancy is not so high and may be justified, at least in part, by accounting for cracking of concrete in the actual structure. However, for low values of c/l, which applies to flat plates, the discrepancies are too high to be attributed solely to reduction of stiffness due to cracking. The low value of k_{ec} is primarily due to low values of k_t calculated for the torsional element as defined by the Code. A balanced improvement for all values of c/l can be achieved if the width of the torsional element as defined in the Code is increased (say by a multiple of slab thickness). This is in accord with the actual behavior of a slab, in which for a concentrated applied moment ($c/l = 0$) a torsional band develops in the transverse direction which has a nonzero width, contrary to the Code recommendation.

CONCLUSIONS

An analysis is proposed for the evaluation of stiffness coefficients of slabs in multiple slab floors resting on an orthogonal grid of columns. The stiffnesses evaluated herein may be used in the equivalent frame method recommended by ACI 318-71 or other appropriate elastic analyses. The analysis is simplified by making proper assumptions for arriving at a lower and an upper bound limit to the slab stiffness. The actual stiffness lying between the two limits proposed. The case of a square panel with several boundary conditions is investigated for a number of column and slab dimensions.

A new set of data is offered for the lower and upper bound stiffnesses of square slabs. For the lower bound, the presented solutions agree closely with the available data. For the upper bound limit, no corresponding solutions seem to have appeared in the literature.

A comparison of the results with ACI 318-71 shows that the latter grossly underestimates the stiffness of the slab obtained from elastic analysis. This is especially more pronounced for lower values of c/l. Keeping the present approach of the Code, a balanced agreement between the Code and the results of elastic plate analysis may be achieved if the width of the torsional beam as defined in the Code is increased (say by some multiple of slab thickness). It should be noted that reduction in slab stiffness due to cracking can, at least partially (depending on the value of c/l and percent and distribution of reinforcement), account for the lower values of stiffness recommended in the Code.

Furthermore, the results illustrate that, for the range of dimensions investigated, the boundary conditions parallel to the framing considered do not significantly affect the slab stiffness and may be disregarded in arriving at a slab stiffness for the equivalent frame method.

To complete the picture in the moment-rotation behavior of concrete slabs, a useful extension of the present work is to consider the case of rectangular panels. This, however, has not been the purpose of the present paper.

NUMERICAL EXAMPLE

Consider a typical square unit of a multiple slab floor 8 in. (20.3 cm) thick and 240 x 240 in. (610 x 610 cm) sides around a square column (capital) 24 x 24 in. (61.0 x 61.0 cm). The slab stiffness for the equivalent frame method under side sway loading is as follows ($\nu = 0$):

$$c = 24 \text{ in. } (61.0 \text{ cm})$$
$$l = 240 \text{ in. } (610 \text{ cm})$$

hence $\dfrac{c}{l} = \dfrac{24}{240} = 0.1$

From the results offered by the present analysis (Fig. 5).

$$k_s \text{ (lower bound)} = 5.52$$
$$k_s \text{ (upper bound)} = 8.47$$

From ACI 318-63, Eq. (1)

$$\bar{k}_s = 12\left(\frac{1}{1-0.1}\right)^3 = 16.45$$

From ACI 318-71

$$\frac{h}{l} = \frac{8}{240} = \frac{1}{30}$$

hence, stiffness coefficient of the torsional strip, Eq. (6)

$$k_t = \frac{72 \times 0.1}{(1-0.1)^3}\left(1 - \frac{0.63}{0.1 \times 30}\right) = 7.80$$

Stiffness coefficient of slab, Eq. (2)

$$k_s = 4 \div \left[\frac{(1-0.1)^3}{3} + (1-0.1)^3\, 0.1 \right.$$
$$\left. \times \left(1 - \frac{0.1}{2}\right) + \frac{(0.1)^2}{2}(1-0.1)^2\left(1 - \frac{0.1}{3}\right)\right]$$
$$= 12.65$$

The equivalent stiffness coefficient of slab, Eq. (4)

$$\frac{1}{k_{ec}} = \frac{1}{7.80} + \frac{1}{12.65}$$

$$k_{ec} = 4.93$$

The elastic stiffness coefficient is between 5.52 and 8.47, more likely closer to 8.47, as compared to 4.93 obtained from the Code.

APPENDIX

Notation

c	=	size of square or equivalent square column, capital or bracket
D	=	flexural rigidity of slab per unit length, $D = Eh^3/[12(1 - \nu^2)]$
E	=	modulus of elasticity for slab concrete
h	=	overall thickness of slab
I	=	moment of inertia of slab beam
k_{es}	=	equivalent stiffness coefficient of slab, k_{es}/D
k_m	=	coefficient for maximum moment in slab, $k_m = M_s/(M_t/c)$
k_s	=	flexural stiffness coefficient of slab, $k_s = K_s/D$
k_s (lower bound), k_s (upper bound)	=	lower and upper bound flexural stiffness coefficients of slab from the present analysis $= M_t/D\theta$
$\overline{k_s}$	=	flexural stiffness coefficient of slab as evaluated from ACI 318-63, $\overline{k_s} = M_t/D\theta$
k_t	=	torsional stiffness coefficient of torsional member, $k_t = K_t/D$
K_{es}	=	equivalent stiffness of slab as defined by Eq. (3)
K_s	=	flexural stiffness of slab (ACI 318-71); moment per unit rotation
K_t	=	torsional stiffness of torsional member (ACI 318-71); moment per unit rotation
l	=	length of span measured center to center of supports
M_t	=	total applied moment at column-slab interface
θ	=	rotation of column in the direction of applied moment M_t
ν	=	Poisson's ratio

ACCURACY OF THE FINITE ELEMENT SOLUTIONS

The data offered in the text of this paper were obtained from a finite element solution using a mesh of 12 x 12 rectangular elements for a quarter of the plate.

To investigate the accuracy of the results and the mode of convergence obtained with mesh refinement, solutions were also obtained using 8 x 8 and 4 x 4 elements for a typical case of $c/l = 0.10$. The finite element mesh, for all cases, was graded with smaller elements within and in the vicinity of the column-slab junction, where high stress gradients exist. The applied moment to the quarter slab was represented by a triangularly distributed load which, in turn, was converted to an equivalent *consistent* load vector.

Fig. A1 shows the lower and upper bound slab flexibilities as a function of mesh size used, for two extreme boundary conditions of simple-free (SF) and clamped-clamped (CC). It is observed that the convergence for displacements are monotonically increasing with mesh refinement and the extrapolated values for the limiting case of infinite number of elements are all within about three percent of solutions for the 12 x 12 mesh used. These results substantiate those of Reference 9 regarding the adequacy of the element type and mesh used in this investigation.

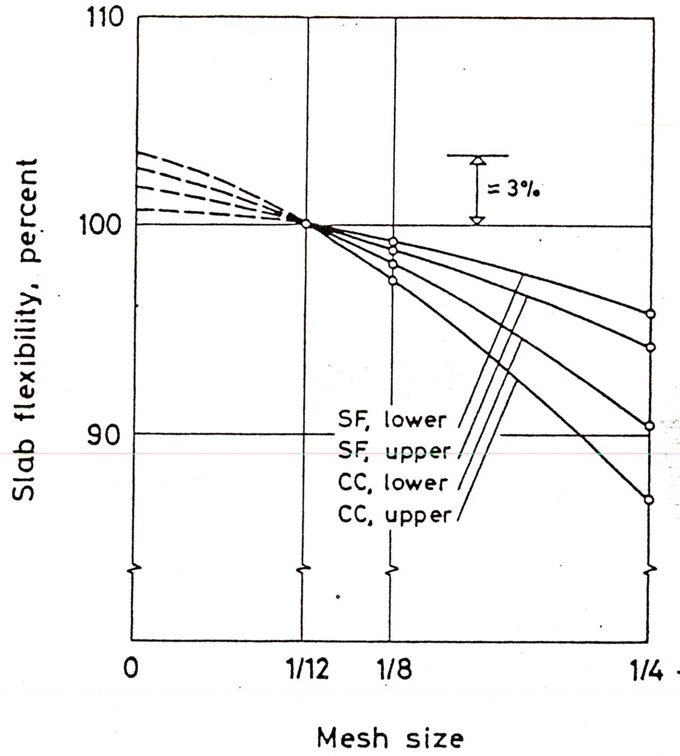

Fig. A1—Convergence and accuracy of the present finite element solution, slab rotational flexibility as percent of values from 12 x 12 mesh

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